Quantitative Finance Problems
Admissions Examination for 2005 Entry
Master of Science in Quantitative Finance

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1 General

1. Compute $\sqrt{645}$ up to 4 decimal places only using a pencil and a piece of paper.

2. Five pirates have 100 gold coins. They have to divide up the loot. In order of seniority (suppose pirate 5 is most senior, pirate 1 is least senior), the most senior pirate proposes a distribution of the loot. They vote and if at least 50% accept the proposal, the loot is divided as proposed. Otherwise the most senior pirate is executed, and they start over again with the next senior pirate. What solution does the most senior pirate propose? Assume they are very intelligent and extremely greedy (and that they would prefer not to die).

3. Given two fuses that burn up within exactly one hour and a fire lighter. The speed of the fuses burning up can vary and is not necessarily constant. Determine a strategy that tells you when 45 Minutes are over.

4. Given nine candidates running for the mayor of Perpignan and a traditional weighing scale that can only compare two different weights. All of the nine candidates are equally heavy except for one who weighs slightly more. Determine the heaviest of the candidates by using the scale at most two times.

5. Fill in the remaining numbers of the magical square, where the sum of three numbers in all rows, all columns and both diagonals is the same.

15	9	
	14	

2 Financial

1. In a standard annuity of n months, total loan amount K, monthly payments, the amount A paid back to the bank every month is constant and is the sum of the interest payment and the amortization. Payments happen at the end of each month, and the first payment is at the end of the first month. Clearly, as time passes the amortization rises. Assuming annual interest rate r compute the remaining debt R after n months. Furthermore, setting $R = 0$, find n. You may assume the time of one month being $1/12$ of a year and the monthly interest rate to be $R/12$. You may check your calculations at http://www.mathfinance.de/annuity.html.

2. The price of an ounce of Gold is quoted in USD. If the price of Gold drops by 5%, but the price of Gold in EUR remains constant, determine the change of the EUR-USD exchange rate.

3. Given the following market of bonds

Bond	Tenor in Years	Price	Notional	Coupon
ZeroBond 1	1	94	100	0%
ZeroBond 2	2	88	100	0%
CouponBond 1	3	100	100	7%
CouponBond 2	4	100	100	8%

write down the cash flows of the four bonds and determine the present value of the cash flow

t_0	1	2	3	4	years
	375	275	575	540	USD

in this market using both a replicating portfolio and a calculation of discount factors.

3 Calculus

1. Does $\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}$ diverge? If so, why? If not, what does it converge to?

2. Determine an explicit formula for the Fibonacci numbers defined recursively by $a_0 = 1$, $a_1 = 1$, $a_{n+2} = a_n + a_{n+1}$.

3. Evaluate the following integrals.

   (a) $\int_0^x \frac{d\,t}{t^2+8t+15}$

   (b) $\int e^x \cos(x)\, dx$

   (c) $\int_0^x \frac{t^2 d\,t}{1+t^3}$

4. Let $n(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ be the density of a standard normal random variable and $\mathcal{N}(x) = \int_{-\infty}^x n(t)\, dt$ its cumulative distribution function. Find $\frac{d}{dx} N^{-1}(x)$ and $\int N(x)\, dx$

5. Find the three second partial derivatives of $f(x,y) = e^{-(x^2+y^2)}$.

6. The function $f(x,y)$ is defined so that $f(0,0) = 0$ and otherwise

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

   Explain carefully whether f

   - is continuous at $(0,0)$,
   - has partial derivatives at $(0,0)$,
   - is differentiable at $(0,0)$.

7. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

   By assuming a solution of the form $u(x,t) = X(x)T(t)$, deduce that

$$u_\alpha(x,t) = (A_\alpha \cos(\alpha x) + B_\alpha \sin(\alpha x))e^{-\alpha^2 t},$$

   where A_α and B_α are constants, is a solution for any constant α. Show that if we now impose the boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(\pi,t) = 0$, then this reduces the possible solutions to those of the form

$$u_n(x,t) = B_n \cos(nx)e^{-n^2 t}$$

   for $n = 0, 1, 2, \ldots$ and where B_n is constant. Hence or otherwise find the solution of the problem with additional initial condition $u(x,0) = 2\cos^5(x), 0 < x < \pi$.

8. Find all eigenvalues and all normalized eigenvectors of the following matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

9. solve the following ordinary differential equations with given initial value.
   - $y' + y\sin(x) = 0, y(\pi) = \frac{1}{e}$
   - $y' + \frac{y}{x} = \frac{\ln x}{x}, y(1) = 1$
   - $xy' = y + \sqrt{x^2 + y^2}$
   - $y'' + 10y' + 21y = 0, y(0) = 0, y'(0) = 4$
   - $y'' - 3y' + 2y = e^{17x}$

4 Probability

1. There are 2 boxes, one with a red and white ball and other with two red balls; one box is in front of you, but you don't know which one. You pull out a ball and it's a red one. What is the probability that the other ball in that box is also red?

2. What is the expected number of tosses to get two heads in a row for an unbiased coin?

3. The random variable X is distributed uniformly on the interval $[0, 1]$. Find the cumulative distribution function and the probability density function of
$$Y = \frac{aX}{1 - X},$$
where a is a constant. When does $\mathbb{E}[Y]$ exist?

4. Let X_1, X_2, X_3, X_4 be independently and identically distributed, with the uniform distribution on $[0, 1]$. Find the cumulative distribution function and the probability density function of
$$Z = \max(X_1, X_2, X_3, X_4).$$
Explain briefly how this result would generalize to the maximum of n such i.i.d. variables.

5. Let X_i be independent and identically distributed Bernoulli random variables with $\mathbb{P}[X_i = 1] = p$ and $\mathbb{P}[X_i = 0] = q = 1 - p$. What is the distribution of
$$Z = X_1 + X_2 + X_3 + \ldots + X_N,$$
when

   (a) $N = n$ is a constant integer,

   (b) N is a random variable with a Poisson distribution with parameter λ, and N is independent of $X_1, X_2, \ldots$,

   (c) N is the score arising from a single throw of a standard (six-faced and fair) die, and N is independent of $X_1, X_2, \ldots$.

   Calculate the expectation of Z for each of these three cases.

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Show that the estimators for the mean and variance are unbiased, i.e. show that for $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$

$$\begin{aligned} \mathbb{E}[\bar{X}] &= \mu, \\ \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2\right] &= \sigma^2. \end{aligned}$$