Credit dynamics in a first passage time model with jumps

Natalie Packham*, Lutz Schlögl† and Wolfgang M. Schmidt‡

September 2009

Abstract

The payoff of many credit derivatives depends on the level of credit spreads. In particular, the payoff of credit derivatives with a leverage component is sensitive to jumps in the underlying credit spreads. In the framework of first passage time models we extend the model introduced in [Overbeck and Schmidt, 2005] to address these issues. In the extended a model, a credit quality process is driven by an Itô integral with respect to a Brownian motion with stochastic volatility. Using a representation of the credit quality process as a time-changed Brownian motion, we derive formulas for conditional default probabilities and credit spreads. An example for a volatility process is the square root of a Lévy-driven Ornstein-Uhlenbeck process. We show that jumps in the volatility translate into jumps in credit spreads. We examine the dynamics of the OS-model and the extended model and provide examples.

Keywords: gap risk, credit spreads, credit dynamics, first passage time models, Lévy processes, general Ornstein-Uhlenbeck processes

JEL classification: G12, G13, G24, C69

MSC 2000 codes: 60G35, 60H05, 91B28

1 Introduction

Other than being subject exclusively to default risk, the payoff of some credit derivatives is determined explicitly by the level of CDS spreads. In this case, the dynamics of CDS spreads play a significant role in product valuation. Examples of such products are default swaptions and credit derivatives with a leverage component. The latter are in addition sensitive to gap risk, a risk that is linked to the occurrence of jumps in the evolution of credit spreads, even if such jumps do not lead to default. The model introduced here is suitable for valuing such credit derivatives.

To motivate this further, let us gather some stylised facts on credit spreads and let us consider the leveraged credit-linked note (LCLN) as an example of a product that features gap risk. We need the notion of a *credit default swap (CDS)* first. Given an underlying entity, such as a company, a CDS is a contract between two counterparties, the protection buyer and the protection seller, that insures the protection buyer against the default event (i.e., failure to fulfill a financial obligation) of the underlying entity. The protection buyer regularly pays a constant premium, the *credit spread (or CDS spread)*, that is fixed at inception, up until maturity of the CDS or the default event, whichever occurs first. This stream of payments is termed the *premium leg* of the CDS. In return, the protection seller

*Natalie Packham, Centre for Practical Quantitative Finance, Frankfurt School of Finance & Management, Sonnemannstr. 9-11, 60314 Frankfurt am Main. n.packham@frankfurt-school.de

†Lutz Schlögl, Fixed Income Quantitative Research, Nomura International Plc, lutz.schloegl@nomura.com

‡Wolfgang M. Schmidt, Centre for Practical Quantitative Finance, Frankfurt School of Finance & Management, Sonnemannstr. 9-11, 60314 Frankfurt am Main. w.schmidt@frankfurt-school.de

agrees to compensate the protection buyer for the loss incurred by default of the underlying entity at the time of default in case this occurs before maturity. This constitutes the *protection leg* of the CDS.

Following the principle of no-arbitrage, the value of a financial claim is considered to be the discounted expectation of its payoff under a risk-neutral measure. The *fair credit spread (or fair CDS spread)* is the CDS spread that makes the value of both legs equal. Since we only consider fair credit spreads in this paper, we just speak of credit spreads. The mapping of CDS spreads with respect to their maturity is called the *term structure of CDS spreads*.

CDS spreads and the term structure of credit spreads change over time. In general, two components of credit risk contribute to the behaviour of CDS spreads: jump-to-default risk and the risk of credit quality changes. We state some stylised facts about the shape and dynamics of the term structure. A wide variety of term structure shapes has been observed in the market, such as upward sloping, flat, hump-shaped and downward sloping curves [Helwege and Turner, 1999], [Zhou, 2001]. Another common observation is that credit spreads do not tend to zero as maturity tends to zero, but are significantly greater than zero [Duffie and Lando, 2001], [Zhou, 2001], [Duffie and Singleton, 2003] and [Lando, 2004]. This indicates that, for any time to maturity, market participants presume a positive probability of unexpected and instantaneous default.

There is also a significant amount of research that indicates that credit spreads are subject to sudden and unexpected jumps [Johannes, 2000], [Das, 2002], [Dai and Singleton, 2003], [Tauchen and Zhou, 2006] and [Zhang et al., 2009]. According to [Zhou, 2001], a more volatile jump component indicates that a firm is more likely to default on its short-maturity bonds than a firm with a more volatile diffusion component. In a recent empirical study on credit spreads, [Schneider et al., 2007] observe that CDS spreads exhibit frequent positive jumps, which typically affect CDS spreads of all maturities. These jumps are attributed to the arrival of bad news. Good news also affect the whole maturity spectrum, but tend to propagate gradually.

As an example of a credit derivative whose payoff is sensitive to the level of credit spreads, consider the *leveraged credit-linked note*. This note is particularly sensitive to jumps in CDS spreads, even if a jump does not lead to default. The principal idea is that an investor sells protection on an amount of default risk that is a multiple k, the *leverage factor*, of his investment amount. The motivation for taking leveraged exposure is to earn a certain multiple $\tilde{k} < k$ of the credit spread. Due to the leverage, his investment will most likely not suffice to compensate the loss incurred by default. Therefore, a trigger is agreed to terminate the structure while the cost of closing the position is still likely to be sufficiently covered by the investment amount. The time and cost of closing the position depends on the level of credit spreads, hence the investor is exposed mainly to spread risk, and to default risk only to a lesser extent. Furthermore, if a sufficiently high jump in the credit spread occurs, the cost of unwinding the transaction may exceed the investment amount, in which case the issuer of the note has to cover the missing amount required to close the position. This type of risk is called *gap risk*. Essentially, valuation of the note means determining the factor $(k - \tilde{k})$ that determines the spread received by the issuer for bearing gap risk.

There are generally two approaches to modelling credit risk: the structural and the reduced-form approach. In reduced-form models, default is not linked to economic variables, but is an unpredictable Poisson-type event, and the main object of the modeller's attention is the hazard rate of the jump process describing default. This approach has been overwhelmingly popular with practitioners, its main advantage being its tractability: It is generally straightforward to fit a given term structure of CDS spreads and the techniques are very similar to those of interest rate modelling. The literature on this type of models is vast, the papers by [Jarrow and Turnbull, 1995], [Lando, 1998] and [Duffie and Singleton, 1999] are only a few classic examples.

From the point of view of spread dynamics, modelling the default time as a totally unpredictable stopping time is not entirely satisfactory. Even with a low initial hazard rate, such a model will assign a non-negligible probability to the possibility of the credit defaulting without a prior movement in the credit spread. Defaults of this type are very uncommon in practice. The default swap market is efficient enough that default events are almost always preceded by a significant widening of credit spreads. It is this spread widening that is the real jump event that a market participant needs to worry about.

The ability of a model to assign probability mass to spread widening scenarios is constrained by the probability assigned to defaults in low spread scenarios, as the model must fit the initial credit spread. From a practical point of view, failure to assign enough probability to spread widening scenarios can lead to a dangerous underpricing of credit spread gap risk. The phenomenology of default that we are trying to capture is the following: a credit with a low default swap spread does not default "out of the blue", but rather some kind of "regime switch" takes place, causing the credit spread to widen, after which the credit may either default or eventually recover.

We implement this idea via a first passage time model where a *credit quality process* exhibits stochastic volatility. In fact, the volatility process is a Levy-driven Ornstein-Uhlenbeck process. A jump in the volatility process is the "regime switch" we alluded to earlier. The current trend is to interpret the class of structural models in a wide sense to include any model where default is modelled as the first hitting time of a certain threshold by an abstract observable credit quality process. In this sense our model is structural, though the term "first passage time model" is technically more accurate. The structural approach, pioneered by [Merton, 1974], has been developed by [Black and Cox, 1976], [Longstaff and Schwartz, 1995], and many others.

As the name suggests, in a first passage time model, the computation of default probabilities is equivalent to computing the distributions of first passage times. The simplest case is that of a Brownian motion hitting a constant barrier, where a simple closed-form solution exists. This simple set-up, however, does not allow one to fit a given term structure of credit spreads. One approach to this problem in the past has been to use a time-dependent barrier, but already in the Brownian motion case this leads to tractability problems. Two further criticisms can be made of first passage time models in the simple diffusion case. Credit spreads only diffuse, whereas in reality they exhibit strong jump dynamics. Short term credit spreads in the model tend to zero, as a diffusion process needs a finite time to cross a barrier. [Zhou, 2001] proposes a model that overcomes both problems by modelling the credit quality process as a jump-diffusion. Computing first passage times for jump-diffusions is a very intractable problem, making the model by [Zhou, 2001] and a generalization by [Kiesel and Scherer, 2007] computationally very demanding. [Baxter, 2007] and [Cariboni and Schoutens, 2007] propose models where the credit quality process is a Lévy process. Again, the first passage time problem is difficult in this context.

[Overbeck and Schmidt, 2005] propose a simple solution to the problem of calibrating a first passage time model to a term structure of credit spreads. Instead of considering the hitting time of Brownian motion to a complicated, time-dependent barrier, they consider the first hitting time of a time-changed Brownian motion to a constant barrier. The time change is continuous, strictly increasing and deterministic. Because both the time change and the underlying Brownian motion are continuous, one can easily adapt the analytic formula from the simple Brownian case and obtain an analytic calibration to a term structure of default probabilities.

Our model builds directly on [Overbeck and Schmidt, 2005]. We consider a credit quality process X to be a time-changed Brownian motion with a stochastic continuous and strictly increasing time change independent of the Brownian motion. Our credit quality process X can also be represented as an Itô integral $X = \int_0^{\cdot} \sigma_u dW_u$ with a Brownian motion W and a càdlàg process σ. Our standard example for the volatility process σ is the square root of an Ornstein-Uhlenbeck process driven by a subordinator, that is, a Lévy process with strictly increasing paths. In the time-changed Brownian motion interpretation, we can write X as $X_t = B_{\Lambda_t}$ with B a Brownian motion and $\Lambda_t = \int_0^t \sigma_u^2 du$. Because the time change Λ is continuous and independent of B, we retain all the tractability of the deterministic case, while jumps in the volatility σ induce jumps in the credit spreads, even though the credit quality process X is continuous. This extended model is both tractable and at the same time allows for meaningful CDS spread dynamics. Furthermore, even though short-term credit spreads tend to zero (a problem which we consider to be secondary to achieving meaningful dynamics in a tractable manner), jump-to-default events can be approximated by introducing large jumps in the volatility process.

In a related paper [Packham et al., 2009], we show that the model can be calibrated to a wide set of term structures and that its implementation allows for efficient valuation of spread-risky claims. In this paper, we also give valuation examples for a leveraged credit-linked note. The structure of the time-changed Brownian motion may also prove powerful when valuing products that depend on the joint dynamics of several correlated underlyings. This has yet to be investigated.

The paper is structured as follows: We introduce some notation that is used throughout in Section 2. The Overbeck-Schmidt model and some of its properties are examined in Section 3. The extended model is introduced in Section 4. In Section 5, we show that jumps in the stochastic volatility of the credit quality process translate into jumps in default probabilities and credit spreads. Furthermore, we examine the dynamics by computing the distributions of conditional default probabilities and the averaged hazard rate (as a proxy for credit spreads). Some examples, including limiting cases, are also presented in this Section.

A result that is needed for computing conditional default probabilities, but that is not specific to the context of finance, is derived in Appendix B. Here, we are concerned with a particular class of continuous local martingales, called *Ocone martingales*, whose representation as a time-changed Brownian motion (by the famous Dambis, Dubins-Schwarz Theorem) features independent Brownian motion and time-change.

2 Notation and preliminaries

Throughout, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e., a family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s \leq t$. The filtration represents the information available in the market, i.e., $\mathcal{F}_t$ contains all events that are observable until time t. In particular, $\mathcal{F}_0$ is $\mathbf{P}$-trivial, i.e., for any $A \in \mathcal{F}_0$, either $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$. We assume that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual hypotheses, i.e., $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. We also assume that the probability space is rich enough to support any objects that we define. If not otherwise stated, all processes are $(\mathcal{F}_t)_{t\geq 0}$-adapted. An $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion W is a Brownian motion W that is $(\mathcal{F}_t)_{t\geq 0}$-adapted and such that for any $s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$.

We omit explicit reference to the underlying entity when we speak of a default event, default time, etc.. Denote by τ the random time of the default event. In our setup, τ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. The distribution function of τ conditional on the information flow $(\mathcal{F}_t)_{t\geq 0}$ is denoted by $P(t,T) := \mathbf{P}(\tau \leq T|\mathcal{F}_t)$. For fixed maturity T, $(P(t,T))_{t\leq T}$ is a process whose dynamics are driven by the information flow $(\mathcal{F}_t)_{t\geq 0}$. For each T, the process $(P(t,T))_{t\leq T}$ is a martingale with a càdlàg modification. Furthermore, $\mathbf{P}(\cdot|\mathcal{F}_t)$ has a version that is a regular conditional probability, so we may treat $\mathbf{P}(\cdot|\mathcal{F}_t)(\omega)$ like a probability measure for $\mathbf{P}$-almost all $\omega \in \Omega$.

We assume that $\mathbf{P}$ is a risk-neutral measure (presupposing existence of a risk-neutral measure), that is, $\mathbf{P}$ is a probability measure equivalent to the real-world probability measure and such that discounted prices are $\mathbf{P}$-martingales. It follows that the market is free of arbitrage.

Let $s(t,T)$ be the fair credit spread at time t of a CDS with maturity T. Then $(s(t,T))_{t\leq T}$ is the spread process with maturity T and $(s(t,T+t))_{t\geq 0}$ is the spread process with time to maturity T.

The link between conditional default probabilities and credit spreads is as follows: Let r be the default-free interest rate, assumed to be constant for simplicity.[1] Furthermore, assume that the payment at default is a fraction $(1-R)$ of the notational amount, $R \in [0,1)$. Entering into a CDS involves no initial cash-flow, that is, the market value of a CDS at inception is 0; in other words, the discounted fair values of the premium and the default legs are equal. From these considerations, one can show that the *fair credit spread* or *fair CDS spread* $s(t,T)$ at time t is given by

$$\begin{aligned}\frac{s(t,T)}{1-R} &= \frac{\int_t^T \mathbf{e}^{-r(u-t)}\,\mathrm{d}P(t,u)}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u} \\ &= \frac{\mathbf{e}^{-r(T-t)}\,P(t,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t,u)\,\mathrm{d}u}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u}, \end{aligned} \tag{1}$$

where the second line is obtained by integration by parts. On $\{\tau \leq t\}$ or for $t \geq T$, we set $s(t,T) = 0$.

3 Overbeck-Schmidt model

The model we propose is a generalisation of the Overbeck-Schmidt model (OS-model) [Overbeck and Schmidt, 2005], which we now introduce. The OS-model allows for straightforward analytic calibration to a given term structure of default probabilities. Although the model exhibits dynamics, these are fully determined by calibration to a given term structure of credit spreads. Our goal will be to extend the OS-model to allow for meaningful dynamics. As a precursor to the analysis of the more general model, we explore some characteristics of the OS-model dynamics. For a thorough analysis of the dynamics of the OS-model, see [Kammer, 2007] and [Packham, 2009].

3.1 Model specification

In the OS-model, the default time τ is determined as the first time that a *credit quality process* $X = (X_t)_{t\geq 0}$ hits a barrier $b < X_0$, that is,

$$\tau = \inf\{t \geq 0 : X_t \leq b\}.$$

The principal idea is to model X as a time-changed Brownian motion. Given a Brownian motion B and a deterministic, strictly increasing and continuous time transformation $\Lambda = (\Lambda_t)_{t\geq 0}$, with $\Lambda_0 = 0$, set

$$X_t := B_{\Lambda_t}, \quad t \geq 0.$$

Assume given the distribution $F(t)$, $t \geq 0$, of the default time. If the time-change Λ is given by

$$\Lambda_t = \left(\frac{b}{\mathrm{N}^{(-1)}\left(\frac{F(t)}{2}\right)} \right)^2, \quad t \geq 0, \tag{2}$$

[1] It is straightforward to extend the setting to a non-constant interest rate by assuming that the short rate r and the default indicator process $(\mathbf{1}_{\{t\leq\tau\}})_{t\geq 0}$ are conditionally independent given $\mathcal{F}_t$, for any $t \geq 0$.

where $\mathrm{N}^{(-1)}$ denotes the inverse of the Normal distribution function, then τ admits the distribution F. Furthermore, if the distribution of τ admits a density, then the time-change is absolutely continuous, and

$$\Lambda_t = \int_0^t \sigma_s^2 \,\mathrm{d}s, \tag{3}$$

with σ a nonnegative square-integrable function. The quadratic variation of X is just $[X, X] = \Lambda$, so that there exists a representation of X as a stochastic integral

$$X_t = \int_0^t \sigma_s \,\mathrm{d}W_s,$$

for some Brownian motion W (cf. Theorem 3.4.2 of [Karatzas and Shreve, 1998]). The volatility σ can be interpreted as the *default speed* in the sense that the higher the default speed, the higher the likelihood of crossing the default barrier.

Overbeck and Schmidt use a multivariate extension of the model to value multi-name credit derivatives, such as first-to-default credit baskets. The payoff of these products does not depend on the level of credit spreads, only on whether default takes place or not. Nonetheless, by specification of the credit quality process, the model exhibits dynamics. These dynamics are fully determined by calibration to market-given default probabilities; it is not possible to assign different dynamics to the same term structure of default probabilities.

In the following, we explore some properties of the OS-model to gain some insight on how to extend the model to account for jumps in the evolution of default probabilities and credit spreads.

3.2 Conditional default probabilities

Recall that $P(t,T) = \mathbf{P}(\tau \leq T|\mathcal{F}_t)$, $T \geq t$, has a version that is a regular distribution function and that$(\mathbf{1}_{\{\tau>t\}}\,\mathbf{P}(\tau \leq T|\mathcal{F}_t))_{t\geq 0}$ is a càdlàg semimartingale. Clearly, τ is an $(\mathcal{F}_t)_{t\geq 0}$-predictable stopping time.

Proposition 3.1. *Let $X = (X_t)_{t\geq 0}$, with $X_t = B_{\Lambda_t}$ and $(\Lambda_t)_{t\geq 0}$ given by Equation (3), and let $\tau = \inf\{t \geq 0 : X_t \leq b\}$, $b < 0$. Then,*

(i) On $\{\tau > t\}$, $P(t,T) = \mathbf{P}(\tau \in (t,T]|X_t)$; in particular, $P(t,T)$ is $\sigma(X_t)$-measurable on $\{\tau > t\}$.

(ii) For $t \leq T$, on $\{\tau > t\}$, the probability of default until T conditional on $\mathcal{F}_t$ is given by

$$P(t,T) = 2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right), \tag{4}$$

where N denotes the standard normal distribution function.

(iii) For any $T \geq 0$, the conditional default process $(P(t,T))_{t\geq 0}$ is continuous in t.

Proof. (i) Observe first that

$$P(t,T) = \mathbf{1}_{\{\tau\leq t\}} + \mathbf{1}_{\{\tau>t\}}\,\mathbf{P}(\tau \in (t,T]|\mathcal{F}_t).$$

The events $\{\tau \in (t,T]\}$ and $\{\min_{t<u\leq T} X_u \leq b\}$ are equivalent, and the claim follows from the Markov property of X.

(ii) The hitting time distribution of a Brownian motion starting at 0 is (see e.g. Section 2.6.A of [Karatzas and Shreve, 1998]),

$$\mathbf{P}\left(\min_{s\leq t} B_s < b\right) = 2\,\mathrm{N}\left(\frac{b}{\sqrt{t}}\right), \quad b < 0. \tag{5}$$

On $\{\tau > t\}$, making use of the fact that X is a Markov process and taking into account that the time-change Λ is continuous,

$$\begin{aligned} P(t,T) &= \mathbf{P}(\tau \in (t,T]|\mathcal{F}_t) = \mathbf{P}\left(\min_{t<u\leq T} X_u \leq b\big|\mathcal{F}_t\right) \\ &= \mathbf{P}\left(\min_{t<u\leq T} X_u \leq b\big|X_t\right) = \mathbf{P}\left(\min_{t<u\leq T} B_{\Lambda_u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\big|B_{\Lambda_t}\right) \\ &= \mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\big|B_{\Lambda_t}\right) = 2\mathrm{N}\left(\frac{b - B_{\Lambda_t}}{\sqrt{\Lambda_T - \Lambda_t}}\right), \end{aligned}$$

where the last step follows since $(B_{\Lambda_t+u} - B_{\Lambda_t})_{u\geq\Lambda_t}$ is a Brownian motion independent of $\mathcal{F}_t$ and $(b - B_{\Lambda_t})$ is $(\mathcal{F}_t)_{t\geq0}$-measurable.

(iii) Taking into account that $\mathrm{N}(\cdot)$, X, $\sqrt{\cdot}$ and Λ are continuous, for any sequence $t_n \to t$, as $n \to \infty$, on $\{\tau > t\}$,

$$\lim_{t_n\to t} P(t_n,T) = \lim_{t_n\to t} 2\mathrm{N}\left(\frac{b - X_{t_n}}{\sqrt{\Lambda_T - \Lambda_{t_n}}}\right) = 2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right) = P(t,T).$$

On $\{\tau < T\}$, to analyse the behaviour around default, observe that by the continuity of X and since $X_\tau = b$, it follows easily that $\lim_{t_n\uparrow\tau} P(t_n,T) = 1$, as $n \to \infty$ by Equation (4).

□

Inspection of Equation (4) reveals that $P(t,T)$ depends on the time-change increment and on the distance of X_t to the barrier. The time-change, being deterministic, has limited impact on the dynamics of $(P(t,T))_{t\geq0}$. The last part of the proposition tells us that it is impossible to generate jumps in a default probability process when the time-change is deterministic and continuous. It is thus clear that the model is unsuitable for valuing gap risk – the probability of a gap event in the OS-model is zero. An important consequence of the proposition is that in the setting of the OS-model a necessary condition for jumps at random times in default probability processes is that the time-change be stochastic.

3.3 Distribution of conditional default probabilities

We compute the distribution of $P(t,T)$. This will be insightful when we compare the OS-model with the extended model later.

Proposition 3.2. *For $T \geq 0$ and $t \leq T$, the distribution function of $P(t,T)$ conditional on $\{\tau > t\}$ is*

$$\mathbf{P}\left(P(t,T) \leq x|\tau > t\right) = \frac{\mathbf{P}(P(t,T) \leq x)}{\mathbf{P}(\tau > t)} = \frac{\mathrm{N}\left(h_1(x)\right) - \mathrm{N}\left(h_2(x)\right)}{1 - 2\mathrm{N}(b/\sqrt{\Lambda_t})}, \quad 0 \leq x < 1,$$

with

$$\begin{aligned} h_1(x) &= \frac{\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} - b}{\sqrt{\Lambda_t}} \\ h_2(x) &= \frac{\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} + b}{\sqrt{\Lambda_t}}. \end{aligned}$$

The conditional density of $P(t,T)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{P}(P(t,T) \le x | \tau > t)$$
$$= \frac{\sqrt{\Lambda_T - \Lambda_t}}{2\sqrt{\Lambda_t}\, \mathrm{n}\left(\mathrm{N}^{(-1)}(x/2)\right)\left(1 - 2\mathrm{N}(b/\sqrt{\Lambda_t}\right)} \left(\mathrm{n}(h_1(x)) - \mathrm{n}(h_2(x))\right), \quad 0 < x < 1,$$

where n *denotes the probability density function of a standard normal random variable.*

Proof. Recall from part (i) of Proposition 3.1 that $P(t,T) = \mathbf{1}_{\{\tau \le t\}} + \mathbf{1}_{\{\tau > t\}}\mathbf{P}(\tau \in (t,T] | X_t)$. For $0 \le x < 1$,

$$\{P(t,T) \le x\} = \{\mathbf{P}(\tau \in (t,T] | X_t) \le x, \tau > t\}, \tag{6}$$

which establishes

$$\mathbf{P}(P(t,T) \le x | \tau > t) = \frac{\mathbf{P}(P(t,T) \le x)}{\mathbf{P}(\tau > t)}, \quad x < 1.$$

Using the fact that Λ is continuous,

$$\begin{aligned}
\mathbf{P}\left(P(t,T) \le x\right) &= \mathbf{P}\left(2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right) \le x, \min_{0<s\le t} X_s > b\right) \\
&= \mathbf{P}\left(2\mathrm{N}\left(\frac{b - B_{\Lambda_t}}{\sqrt{\Lambda_T - \Lambda_t}}\right) \le x, \min_{0<s\le \Lambda_t} B_s > b\right) \\
&= \mathbf{P}\left(B_{\Lambda_t} \ge -\left(\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} - b\right), m_{\Lambda_t} > b\right),
\end{aligned}$$

where $m_t = \min_{0<s\le t} B_s$ is the running minimum of B at t. We have (see e.g. Section 2.8.A of [Karatzas and Shreve, 1998])

$$\mathbf{P}(B_t \ge a, m_t \le c) = \mathrm{N}\left(\frac{2c - a}{\sqrt{t}}\right), \quad c \le 0, a \ge c.$$

Continuing above, yields

$$\begin{aligned}
&\mathbf{P}\left(B_{\Lambda_t} \ge -\left(\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} - b\right), m_{\Lambda_t} > b\right) \\
&\quad = \mathbf{P}\left(B_{\Lambda_t} \ge -\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} + b\right) \\
&\qquad - \mathbf{P}\left(B_{\Lambda_t} \ge -\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} + b, m_{\Lambda_t} \le b\right) \\
&\quad = \mathrm{N}\left(\frac{\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} - b}{\sqrt{\Lambda_t}}\right) - \mathrm{N}\left(\frac{\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\sqrt{\Lambda_T - \Lambda_t} + b}{\sqrt{\Lambda_t}}\right).
\end{aligned}$$

The density is obtained by differentiation. □

Examples of conditional default densities are given in the left picture of Figure 1. Here, the conditional density of $\mathbf{P}(\tau \in (1,5) | X_1)$ is shown for initial default distributions $F(T) = 1 - \mathbf{e}^{-hT}$, $T \ge 0$, $h \in \{0.03, 0.05, 0.07\}$. Consider for example the distribution of the default probability with $h = 0.03$. The initial 5-year default probability is approx. 0.14. It turns out that with a probability of roughly 1/2 the default probability in one year is below 0.05, and with a probability of roughly 1/5 it is above 0.2. Loosely speaking, with a high probability, the underlying entity will have either a very high or a very low credit quality in one year. This suggests a very volatile movement of the default probability process, which is explained as follows: In order to match the initial 5-year default probability, the credit

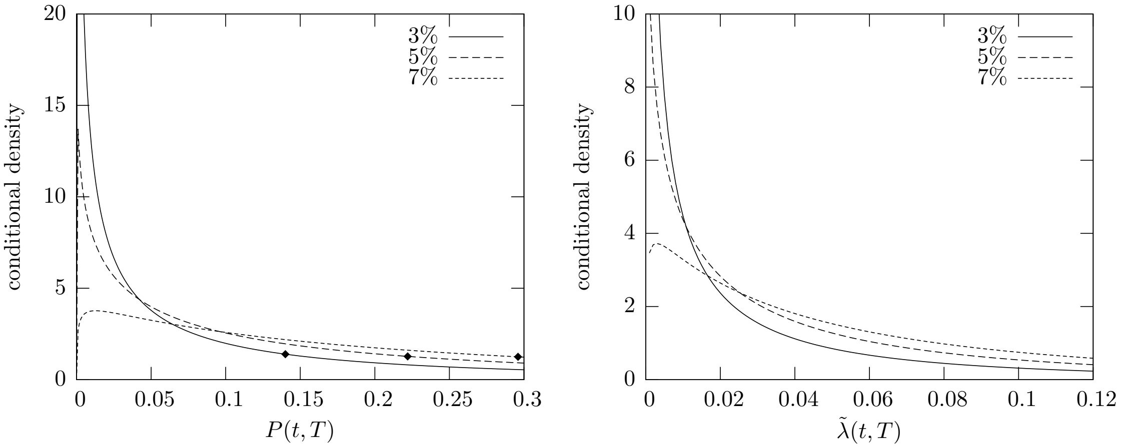

Figure 1: Densities of $P(t,T)$ (left) and $\tilde{\lambda}(t,T)$ (right) conditional on $\{\tau > t\}$. The term hazard rate $\tilde{\lambda}(t,T)$ is an approximation of the credit spread, i.e., $s(t,T) \approx (1-R)\tilde{\lambda}(t,T)\cdot 10^4$ (in basis points). The parameters are $t = 1, T = 5$ and initial hazard rates $3\%, 5\%, 7\%$ (see text). The diamonds mark the initial 5-year default probability $F(T) = \mathbf{P}(\tau \leq T)$. The choice of the barrier is arbitrary, see Remark 3.3.

quality process must adopt a high default speed. The high volatility of the credit quality process then leads to a volatile default probability process.

One can also see (and show) that, as $x \to 0$, the density either converges to 0 or diverges. In fact, for sufficiently small remaining time to maturity, the density diverges. Loosely speaking, this effect can be explained by observing that, as t approaches maturity, the probability mass concentrates at 0 (conditional on no default).

The right picture of Figure 1 shows the conditional density of the term hazard rate $\tilde{\lambda}(t,T)$, a proxy for the credit spread $s(t,T)$. For details on the term hazard rate, see Appendix A. The conditional distribution and density are obtained easily by observing that $P(t,T)$ is a monotone function of $\tilde{\lambda}(t,T)$ (on $\{\tau > t\}$).

Apart from the fact that the dynamics are fully specified by calibration to the given default probability distribution, we may ask how we can obtain distributions that are more realistic. The first issue will be solved by incorporating a stochastic volatility (i.e., stochastic time-change); the second issue will be addressed by incorporating jumps into the stochastic volatility.

Remark 3.3. An interesting point to note is that the dynamics are not influenced by the choice of the barrier b, although at first sight of Proposition 3.2, one may be lead to think so. This follows from the relationship between the barrier and the time transformation. Indeed, using Equation (2), we can re-write h_1 and h_2 as follows: For any $T \geq t$,

$$h_{1,2}(x) = \frac{\mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\left[\left(\frac{b}{\mathrm{N}^{(-1)}(\mathbf{P}(\tau\leq T)/2)}\right)^2 - \left(\frac{b}{\mathrm{N}^{(-1)}(\mathbf{P}(\tau\leq t)/2)}\right)^2\right]^{\frac{1}{2}} \mp b}{\frac{b}{\mathrm{N}^{(-1)}(\mathbf{P}(\tau\leq t)/2)}}$$

$$= \mathrm{N}^{(-1)}\left(\frac{x}{2}\right)\left[\left(\frac{\mathrm{N}^{(-1)}(\mathbf{P}(\tau\leq t)/2)}{\mathrm{N}^{(-1)}(\mathbf{P}(\tau\leq T)/2)}\right)^2 - 1\right]^{\frac{1}{2}} \mp \mathrm{N}^{(-1)}\left(\frac{\mathbf{P}(\tau\leq t)}{2}\right).$$

4 First passage time model with jumps

We have seen in the last section that the Overbeck-Schmidt model is unsuitable for valuing spread products, the principal reason being that the dynamics are completely determined by calibrating the model to a given term structure of default probabilities (or credit spreads). We have further seen that the model is unable to produce random jumps in credit spreads, and that default probabilities and the hazard rate are quite volatile. On the other hand, the OS-model is easy to calibrate and conditional default probabilities are easy to compute.

The natural question is then: Can we extend the OS-model to allow for meaningful dynamics while at the same time maintaining its tractability?

More specifically, in the OS-model, if X is a credit quality process, then on $\{\tau > t\}$ we have $P(t,T) = F(t,T,X_t)$ for some function F by the Markov property of X, and $(P(t,T))_{t\geq 0}$ is mainly driven by the credit quality process X. The dynamics of X are completely determined by calibration to the spot curve, however. We rephrase our question as follows: Can we enrich the dynamics in the sense that $P(t,T) = F(t,T,X_t,Y_t)$ on $\{\tau > t\}$, so that $(P(t,T))_{t\geq 0}$ is driven by a Markov process (X,Y) with X the credit quality process, and Y some other process that produces desired dynamics?

We proceed as follows: The credit quality process X will be defined as a stochastic integral with respect to a Brownian motion, $X = \int_0^\cdot \sigma_s \, \mathrm{d}W_s$, where the integrand σ is a càdlàg stochastic process independent of W. If, for example, σ is the solution to an SDE driven by a Lévy process, then, under some mild conditions, (X,σ) is a Markov process and σ plays the role of the process Y mentioned above. We derive a formula for conditional default probabilities. In a more concrete setting, we specify the volatility process σ as the square root of a Lévy-driven Ornstein-Uhlenbeck process. Typically, in applications, the driving Lévy process will be a compound Poisson process with positive jumps, which reflects the occurrence of jumps as rare events.

4.1 Credit quality process with stochastic volatility

Definition 4.1. The *credit quality process* $X = (X_t)_{t\geq 0}$ of a risky entity is defined to be

$$X_t = \int_0^t \sigma_s \, \mathrm{d}W_s, \quad t \geq 0,$$

where W is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and σ is a strictly positive $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process independent of W with $\mathbf{P}(\int_0^t \sigma_s^2 \, \mathrm{d}s < \infty) = 1$, $t \geq 0$, and $\lim_{t\to\infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$ $\mathbf{P}$–a.s..[2]

To emphasise the association with σ, we speak of X as a *credit quality process with volatility* σ or as a *credit quality process with variance* σ^2. Denote the quadratic variation process of X by $\Lambda = (\Lambda_t)_{t\geq 0}$, with $\Lambda_t = \int_0^t \sigma_s^2 \, \mathrm{d}s$. Observe that Λ is continuous, strictly increasing and $(\mathcal{F}_t)_{t\geq 0}$-adapted.

As before, the default time τ of the risky entity associated with the credit quality process X is the first time that X hits a barrier $b < 0$:

$$\tau = \inf\{t \geq 0 : X_t \leq b\}.$$

Analogous to the OS-model, we can express the credit quality process as a time-changed Brownian motion. Define the family of $(\mathcal{F}_t)_{t\geq 0}$-stopping times $\tau_t = \inf\{s \geq 0 : \Lambda_s > t\}$, $t \geq 0$. By application of the Theorem of Dambis, Dubins-Schwarz (see e.g. [Karatzas and Shreve, 1998, Chapter 3.4.B]) the process B, with $B_t = X_{\tau_t}$, $t \geq 0$, is an $(\mathcal{F}_{\tau_t})$-Brownian motion. Conversely, the credit quality process X can be expressed as a time-changed Brownian motion, $X_t = B_{\Lambda_t}$, $t \geq 0$. We refer to B as the DDS-Brownian motion of X. Furthermore, for every $t > 0$, Λ_t is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-stopping time. Since Λ is continuous, strictly increasing and $\lim_{t\to\infty} \Lambda_t = \infty$ $\mathbf{P}$–a.s., it follows that $\tau_{\Lambda_t} = \Lambda_{\tau_t} = t$. Clearly, the

[2]The requirement $\lim_{t\to\infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$ $\mathbf{P}$–a.s. ensures that $\tau < \infty$ $\mathbf{P}$–a.s., as will become clear later on.

credit quality process model of Definition 4.1 is a generalisation of the OS-model with an absolutely continuous time-change.

To compute conditional default probabilities, we shall use the fact that X is an Ocone martingale, that is, a martingale whose DDS-Brownian motion and quadratic variation (the time-change associated with the DDS-Brownian motion) are independent. That X is an Ocone martingale is proved in Proposition B.2 in Section B. We refer to Section B for a review and some results on time-changes and Ocone martingales.

4.2 Conditional default probabilities

We analyse the properties of conditional default probabilities as we did for the OS-model in Section 3.2. In particular, we are interested in deriving results analogous to Proposition 3.1. As in the OS-model, τ is an $(\mathcal{F}_t)_{t\geq 0}$-predictable stopping time. For an analogue of the last part of Proposition 3.1, which states that the default probability process $(P(t,T))_{t\geq 0}$ cannot jump, we have to be more specific about the volatility process. We shall show in Section 5.1 that, although the credit quality process is continuous, it is possible to specify a model in which default probability processes and credit spreads exhibit jumps. In the following, we state a formula for conditional default probabilities, analogous to Equation (4). Moreover, if (X,σ) is a Markov process, then, on $\{\tau > t\}$, $P(t,T)$ is $\sigma(X_t) \vee \sigma(\sigma_t)$-measurable.

The fact that X is an Ocone martingale, together with Corollary B.5 of Appendix B are the key for establishing a formula for conditional default probabilities $P(t,T) = \mathbf{P}(\tau \leq T|\mathcal{F}_t)$, $t \geq 0$, $T > t$.

Proposition 4.2. *Let X be a credit quality process with volatility process σ. Let $\tau = \inf\{t \geq 0 : X_t \leq b\}$ be the associated default time. On $\{\tau > t\}$, the probability of default until time $T > t$, conditional on $\mathcal{F}_t$, is given by*

$$\mathbf{P}(\tau \leq T|\mathcal{F}_t) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)\middle|\mathcal{F}_t\right) \quad \mathbf{P}\text{–a.s.}. \tag{7}$$

Proof. Let B be the DDS-Brownian motion of X, and recall that $B_{\Lambda_t} = X_t$, $t \geq 0$. By continuity of Λ and by properties of conditional expectation, $\mathbf{P}$–a.s., on $\{\tau > t\}$,

$$\begin{aligned}\mathbf{P}(\tau \leq T|\mathcal{F}_t) &= \mathbf{P}\left(\min_{t<u\leq T} X_u \leq b\middle|\mathcal{F}_t\right) = \mathbf{P}\left(\min_{t<u\leq T} B_{\Lambda_u} \leq b\middle|\mathcal{F}_t\right)\\ &= \mathbf{P}\left(\min_{\Lambda_t<u\leq \Lambda_T} B_u \leq b\middle|\mathcal{F}_t\right) = \mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} \leq b\middle|\mathcal{F}_t\right)\\ &= \mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\middle|\mathcal{F}_t\right)\\ &= \mathbb{E}\left(\mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\middle|\mathcal{F}_t \vee \sigma(\Lambda_T)\right)\middle|\mathcal{F}_t\right). \end{aligned} \tag{8}$$

The random time Λ_t is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-stopping time, and with $\mathcal{F}_{\tau_{\Lambda_t}} = \mathcal{F}_t$ it follows from Corollary B.5 that $(B_{\Lambda_t+u} - B_{\Lambda_t})_{u\geq 0}$ is a Brownian motion independent of $\mathcal{F}_t \vee \sigma(\Lambda_T) \subseteq \mathcal{F}_t \vee \mathcal{F}^\sigma_\infty$. On the other hand, the random variables $\Lambda_T - \Lambda_t$ and $b - B_{\Lambda_t}$ are $\mathcal{F}_t \vee \sigma(\Lambda_T)$-measurable. Hence, by the first passage time distribution of Brownian motion, cf. Equation (5), $\mathbf{P}$–a.s.,

$$\mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\middle|\mathcal{F}_t \vee \sigma(\Lambda_T)\right) = 2\mathrm{N}\left(\frac{b - B_{\Lambda_t}}{\sqrt{\Lambda_T - \Lambda_t}}\right).$$

Inserting into Equation (8) yields Equation (7). □

Remark 4.3. It is easily verified that the conditional default distribution is well-defined, i.e., that the $T \mapsto P(t,T)$ fulfills the properties of a distribution function, for every $t \geq 0$. This follows from the condition that σ be strictly positive, which implies that the time-change Λ is strictly increasing.

The proof of the following corollary is straightforward.

Corollary 4.4. *Let X be a credit quality process with volatility process σ, and assume further that (X, σ) has the Markov property. Let τ be the associated default time. Then, for $T > t$, on $\{\tau > t\}$, the conditional default distribution is*

$$\mathbf{P}(\tau \leq T|\mathcal{F}_t) = \mathbf{P}(\tau \leq T|X_t, \sigma_t) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)\Big|X_t, \sigma_t\right) \quad \mathbf{P}\text{–}a.s.. \tag{9}$$

Finally, by setting $t = 0$ we obtain a formula for unconditional default probabilities:

Corollary 4.5. *Let X be a credit quality process with volatility process σ, and let τ be the associated default time. Then, the default distribution is given by*

$$\mathbf{P}(\tau \leq T) = 2\mathbb{E}\left(\mathrm{N}\left(\frac{b}{\sqrt{\Lambda_T}}\right)\right), \quad T \geq 0. \tag{10}$$

It can also be shown that the conditional default distribution $P(t, T)$, $T \geq t$, is absolutely continuous and has a density, see Section 6.9 of [Packham, 2009].

4.3 Variance as Lévy-driven Ornstein-Uhlenbeck process

We put the model to work by specifying the variance process σ^2 to be a mean-reverting process with jumps. Candidates as drivers for the variance process are Lévy processes: they incorporate jumps, and we can build Markov processes by specifying the dynamics of the variance with respect to Lévy processes.

As an explicit example we model the variance process as a Lévy-driven Ornstein-Uhlenbeck process, which we now introduce.

Definition 4.6. Let $Z = (Z_t)_{t\geq 0}$ be a subordinator, that is, a Lévy process with non-decreasing paths $\mathbf{P}$–a.s.. The process $Y = (Y_t)_{t\geq 0}$, defined by

$$Y_t = Y_0\,\mathbf{e}^{-at} + \int_0^t \mathbf{e}^{-a(t-s)}\,\mathrm{d}Z_s, \quad t \geq 0,\, a \in \mathbb{R},$$

is a *Lévy-driven Ornstein-Uhlenbeck process (LOU process).*

The LOU process Y is the solution of the SDE

$$\mathrm{d}Y_t = -aY_{t-}\,\mathrm{d}t + \mathrm{d}Z_t, \quad t \geq 0.$$

It is easily seen that, if Z is a pure-jump process with positive jumps, then Y moves up by jumps and decays exponentially in-between the jumps. Models where an asset price's variance is driven by an LOU process are considered for example by [Barndorff-Nielsen and Shephard, 2001]. For details on LOU processes, see also [Norberg, 2004], Chapter 5 of [Schoutens, 2003] and Chapter 15.3.3 of [Cont and Tankov, 2004].

We specify the variance process as an LOU process that incorporates in addition a deterministic function of time – denoted by θ below –, which will turn out to be useful for calibration. In the following, when we speak of an LOU process, we mean the more general process given by Equation (12) below.

Proposition 4.7. *Let Z be an $(\mathcal{F}_t)_{t\geq 0}$-subordinator, let $a \in \mathbb{R}_+$ and let θ be a bounded and càdlàg function. Then a solution σ^2 to the SDE*

$$\mathrm{d}\sigma_t^2 = a(\theta(t) - \sigma_{t-}^2)\,\mathrm{d}t + \mathrm{d}Z_t \tag{11}$$

exists and is given by

$$\sigma_t^2 = \mathbf{e}^{-at}\,\sigma_0^2 + \int_0^t \mathbf{e}^{-a(t-u)}\,a\,\theta(u)\,\mathrm{d}u + \int_0^t \mathbf{e}^{-a(t-u)}\,\mathrm{d}Z_u, \quad t \geq 0. \tag{12}$$

Moreover, for any $T > t$, the increment of the integrated process is

$$\int_t^T \sigma_u^2\,\mathrm{d}u = \left(1 - \mathbf{e}^{-a(T-t)}\right)\frac{\sigma_t^2}{a} + \int_t^T \theta(u)\left(1 - \mathbf{e}^{-a(T-u)}\right)\mathrm{d}u + \frac{1}{a}\int_t^T \left(1 - \mathbf{e}^{-a(T-u)}\right)\mathrm{d}Z_u. \tag{13}$$

The credit quality process model with LOU variance process is then:

Proposition 4.8. *Let W be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, and let $\sigma^2 = (\sigma_t^2)_{t\geq 0}$, $\sigma_0^2 > 0$, be as in Proposition 4.7, with Z independent of W and with θ such that $\sigma_t^2 > 0$, $t \geq 0$. Then the stochastic process $X = (X_t)_{t\geq 0}$,*

$$X_t = \int_0^t \sigma_s\,\mathrm{d}W_s, \quad t \geq 0, \tag{14}$$

is a credit quality process in the sense of Definition 4.1. Moreover, (X, σ) is a Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proof. That $\mathbf{P}(\int_0^t \sigma_s^2\,\mathrm{d}s < \infty) = 1$, $t \geq 0$, is a consequence of Equation (13) and the fact that $Z_t < \infty$ $\mathbf{P}$–a.s. for any $t \geq 0$, since Z is a subordinator. Finally, $\lim_{t\to\infty} \int_0^t \sigma_s^2\,\mathrm{d}s = \infty$, since for a subordinator Z, $\lim_{t\to} Z_t = \infty$ $\mathbf{P}$–a.s. (see e.g. Section 3.1 of [Bertoin, 1998]). That (X, σ) is a Markov process follows from Theorem V.32 of [Protter, 2005]. □

A sample path of σ^2 and $\Lambda = \int_0^t \sigma_s^2\,\mathrm{d}s$ is given in Figure 2. Here, Z is a compound Poisson process with positive jumps.

5 Dynamics

In this section we establish some results concerning the dynamics of conditional default probabilities and credit spreads, resp. term hazard rates. We show that in the LOU variance process model, jumps in the variance propagate into jumps in default probabilities and credit spreads. We compute the distribution of conditional default probabilities and the term hazard rate. For specification of the dynamics of default probabilities and credit spreads using the Itô formula we refer to Chapter 7 of [Packham, 2009].

5.1 Jumps in default probabilities and credit spreads

The continuity of the credit quality process X and the associated time-change Λ were essential to derive the formula for conditional default probabilities, Equation (7), from which credit spreads can be computed. But recall that we wished to build a model that incorporates jumps in credit spreads. This is established by the following propositions.

Proposition 5.1. *Let X be a credit quality process with LOU variance process σ^2 as in Proposition 4.8 with σ^2 driven by a subordinator. Let $\tau = \inf\{t > 0 : X_t \leq b\}$ be the associated default time. Fix $T > 0$ and let $(P(t,T))_{t\leq T}$ be the associated conditional default probability process. Then, for $\mathbf{P}$-almost all $\omega \in \{\tau > t\}$, $(P(t,T))_{t\leq T}$ is a process whose jumps are positive and*

$$\Delta\sigma_t^2(\omega) = 0 \iff \Delta P(t,T)(\omega) = 0, \quad \textit{for all } T > t.$$

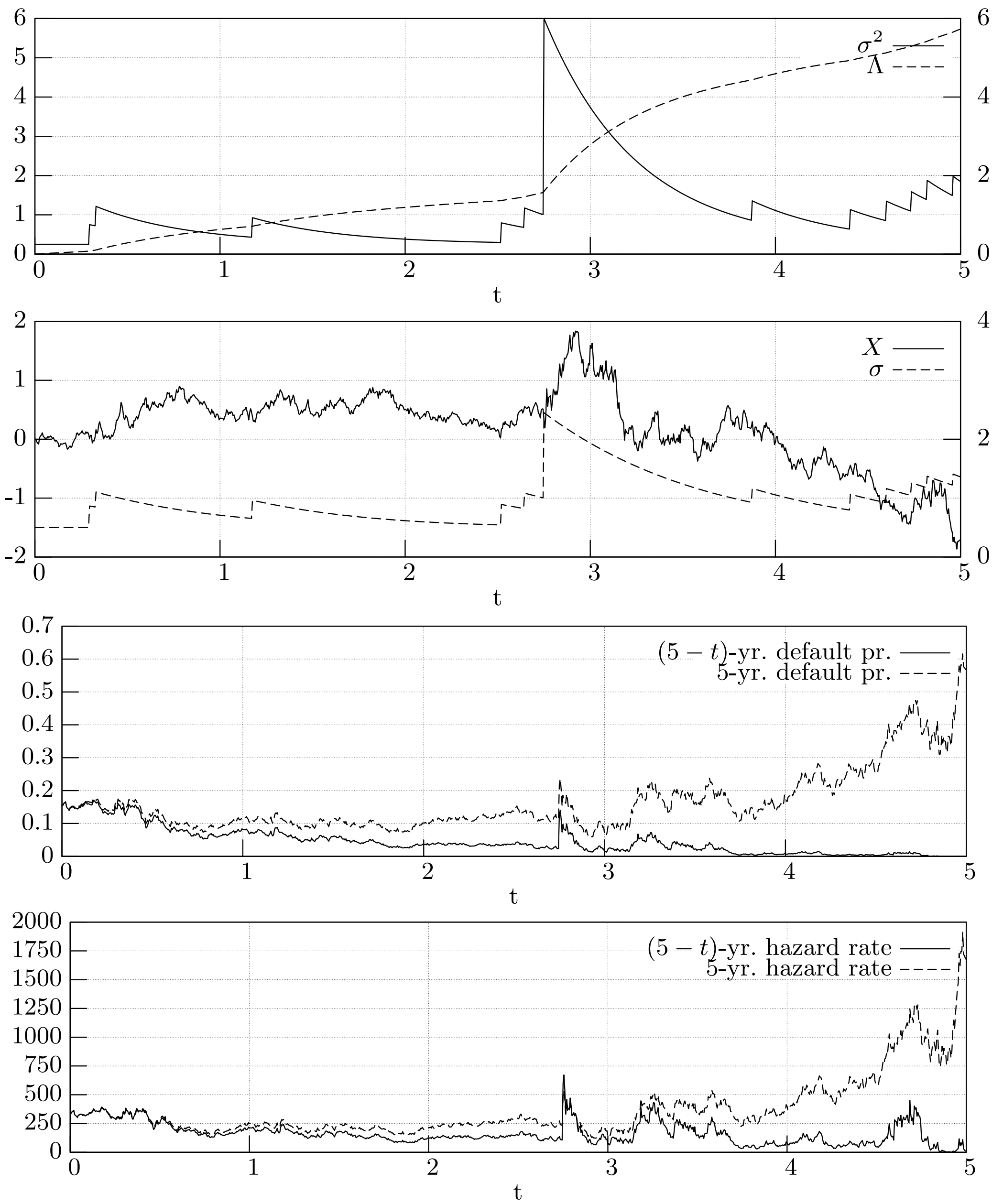

Figure 2: Example of variance process and credit quality process. Top: variance process σ^2, Equation (12) (continuous line, left axis); time-change Λ, Equation (13) (dashed line, right axis).
Second from top: credit quality process X, Equation (14) (continuous line, left axis); volatility σ (dashed line, right axis).
Second from bottom: 5-year default probability process, with decaying time-to-maturity (continuous line) and with fixed time-to-maturity (dashed line).
Bottom: Term hazard rate computed from default probabilities, i.e., $-\ln(1-P(t,T))/(T-t)$ (continuous line), $-\ln(1-P(t,T+t))/T$ (dashed line).
Parameters: $a = 2$, $\theta \equiv 0.25$, $\sigma_0^2 = 0.25$; σ^2 is driven by a compound Poisson process with jump intensity $\lambda = 2$ and discrete jump size distribution with jump sizes $0.05, 5$ with probabilities $0.95, 0.05$, respectively. The barrier is $b = -3$.

Moreover, for any $t < T$, $\mathbf{P}$–a.s.,

$$\Delta P(t,T) = 2\mathbb{E}\left(\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right) \big| X_t, \sigma_t^2\right) - 2\mathbb{E}\left(\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right) \big| X_t, \sigma_{t-}^2\right). \tag{15}$$

Proof. Abbreviate

$$\begin{aligned} h(t,T) &= \int_t^T \theta(u)\left(1 - \mathbf{e}^{-a(t-u)}\right) \mathrm{d}u \\ L_{t,T} &= \int_t^T \left(1 - \mathbf{e}^{-a(T-u)}\right) \mathrm{d}Z_u, \end{aligned}$$

so that, by Equation (13),

$$\Lambda_T - \Lambda_t = \left(1 - \mathbf{e}^{-a(T-t)}\right)\frac{\sigma_t^2}{a} + h(t,T) + \frac{L_{t,T}}{a}.$$

On $\{\tau > t\}$ and by the Markov property of (X, σ), the conditional default probability at t until T is given by Equation (9), and by the independence of $L_{t,T}$ and (X, σ), a version of this conditional probability is given by $g_{t,T}(X_t, \sigma_t)$ with

$$g_{t,T}(x,y) := \mathbb{E}\left(2\mathrm{N}\left(\frac{b - x}{\sqrt{\left(1 - \mathbf{e}^{-a(T-s)}\right) y^2/a + h(t,T) + L_{t,T}/a}}\right)\right). \tag{16}$$

To derive the claim of the Proposition we require the following:

(i) $L_{t-,T} = L_{t,T}$ $\mathbf{P}$–a.s., for any $t \le T$,

(ii) for any sequence $(t_n, x_n, y_n) \to (t, x, y)$,

$$g_{t_n,T}(x_n, y_n) \to g_{t,T}(x,y), \quad \mathbf{P}\text{–a.s.}, \tag{17}$$

(iii) for $(b - x) < 0$, $g_{t,T}(x,y)$ is strictly increasing in y.

Property (i) is well-known, see e.g. [Sato, 1999, p. 6]. For (ii) observe that

$$\frac{b - x_n}{\sqrt{\left(1 - \mathbf{e}^{-a(T-t_n)}\right) y_n^2/a + h(t_n,T) + L_{t_n,T}/a}} \\ \to \frac{b - x}{\sqrt{\left(1 - \mathbf{e}^{-a(T-t)}\right) y^2/a + h(t,T) + L_{t,T}/a}}, \quad \mathbf{P}\text{–a.s.}, \quad \text{as } n \to \infty,$$

by (i) and since all the terms in the sum of the denominator converge and the limit of the denominator is greater 0, for $t < T$. Equation (17) is obtained by continuity of the Normal distribution and Dominated Convergence.

For (iii) observe that the denominator in Equation (16) is strictly increasing in y and that for $(b - x) < 0$, $t \mapsto \mathrm{N}((b - x)/\sqrt{t})$ is strictly increasing.

Fix $g_{t,T}(X_t, \sigma_t)$ as the version of the conditional default probability $P(t,T)$ on $\{\tau > t\}$. Then, taking into account that X is continuous $\mathbf{P}$–a.s., and that on $\{\tau > t\}$ we have $(b - X_t) < 0$, we obtain $\mathbf{P}$–a.s. for any sequence $t_n \uparrow t$,

$$P(t-,T) = \lim_{t_n \uparrow T} g_{t_n,T}(X_{t_n}, \sigma_{t_n}) = g_{t,T}(X_t, \sigma_{t-}) \begin{Bmatrix} = g_{t,T}(X_t, \sigma_t), & \text{if } \Delta\sigma_t = 0 \\ < g_{t,T}(X_t, \sigma_t), & \text{if } \Delta\sigma_t > 0 \end{Bmatrix} = P(t,T).$$

□

A straightforward consequence of this Proposition is that a jump of σ_t^2 triggers a jump in all conditional default probability processes $(P(t,T))_{t\le T}$, $T \ge 0$.

The variance process in the example of Figure 2 has a large jump, which is also clearly visible in the default probability process.

Remark 5.2. Observe that in the proof we have not used properties specific to the mean-reversion feature of the LOU process. In order to establish the statement for other Markov processes (X, σ^2) where σ^2 is driven by a Lévy process, essentially, one must show that a corresponding version of Equation (17) holds.

For CDS spreads we have the following result:

Proposition 5.3. *Let $(s(t,T))_{0\le t\le T}$ be the CDS spread process for maturity T. Then $(s(t,T))_{0\le t\le T}$ is càdlàg, and for $t \le T$ and for $\mathbf{P}$-almost all $\omega \in \{\tau > t\}$,*

$$\Delta s(t,T)(\omega) > 0 \iff (\Delta P(t,u)(\omega) > 0, \text{ for some } u \in (t,T]).$$

Proof. The claim is derived using the relationship between the CDS spread and conditional default probabilities, Equation (1). Consider first the integral of the numerator of Equation (1). For any sequence $t_n \to t$, as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{t_n}^T r\,\mathbf{e}^{-r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u = \underbrace{\lim_{n\to\infty} \int_{t_n}^t r\,\mathbf{e}^{-r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u}_{=0} + \lim_{n\to\infty} \int_{t}^T r\,\mathbf{e}^{-r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u.$$

Then, for the numerator of Equation (1), it follows by Dominated Convergence, for any sequence $t_n \uparrow t$ as $n \to \infty$,

$$\lim_{n\to\infty} \left[\mathbf{e}^{-r(T-t_n)} P(t_n,T) + \int_t^T r\,\mathbf{e}^{r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u\right] = \mathbf{e}^{-r(T-t)} P(t-,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t-,u)\,\mathrm{d}u.$$

Similarly, we obtain for the denominator,

$$\lim_{n\to\infty} \int_{t_n}^T \mathbf{e}^{-r(u-t_n)}\,(1-P(t_n,u))\,\mathrm{d}u = \int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t-,u))\,\mathrm{d}u.$$

It follows that

$$\begin{aligned}\frac{\Delta s(t,T)}{1-R} &= \frac{\mathbf{e}^{-r(T-t)} P(t,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t,u)\,\mathrm{d}u}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u} \\ &\quad - \frac{\mathbf{e}^{-r(T-t)} P(t-,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t-,u)\,\mathrm{d}u}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t-,u))\,\mathrm{d}u} \\ &\begin{cases} = 0, & \text{if } \Delta P(t,u) = 0, \quad t < u \le T \\ > 0, & \text{if } \Delta P(t,u) > 0, \quad t < u \le T. \end{cases}\end{aligned} \tag{18}$$

In particular, this establishes that $\Delta s(t,T) > 0 \Longrightarrow (\Delta P(t,u) > 0, \text{ for some } u \in (t,T])$. Finally, for the converse statement observe that $\Delta P(t,u) > 0$ for some $u \in (t,T]$ if and only if $\Delta P(t,u) > 0$ for all $u \in (t,T]$ by the previous Proposition and by Equation (18). □

In a similar way, noting that Λ is continuous, the corresponding result for CDS spread processes with a fixed time-to-maturity is obtained.

Corollary 5.4. *Let $(s(t,t+T))_{t\geq 0}$ be the CDS spread process for time-to-maturity T. Then, under the assumptions of Proposition 5.3, $(s(t,t+T))_{t\geq 0}$ is càdlàg and for $t \leq T$ and $\mathbf{P}$–a.a. $\omega \in \{\tau > t\}$,*

$$\Delta s(t,t+T)(\omega) > 0 \iff \Delta P(t,u)(\omega) > 0, \text{ for some } u \in (t, T+t].$$

Obviously, the model excludes events where credit spreads jump for selected maturities only. However, this is compatible with the observation that credit spreads tend to jump together.

Remark 5.5. It is easily seen that a jump in the variance process σ^2 cannot lead to default $\mathbf{P}$–a.s.. It suffices to recall that $\tau = \inf\{t > 0 : X_t \leq b\}$ is a predictable stopping time, whereas the jumps of the driving subordinator are totally inaccessible. As a consequence, one can show that the short-term credit spread vanishes, preventing jump-to-default events; see Section 7.2 of [Packham, 2009]. However, we shall see shortly that it is possible to specify the model in such a way that it allows for "near" jump-to-default events.

5.2 Distribution of conditional default probabilities

We compute the distribution of conditional default probabilities as we did for the OS-model in Section 3.3. Fixing $t \geq 0$ and $T \geq t$, the goal is to compute the distribution of $P(t,T)$. Define

$$f_{\sigma_t^2}(X_t) := 2\mathbb{E}\left(\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)\Big|X_t, \sigma_t^2\right).$$

Observe that $f_{\sigma_t^2}$ is continuous and strictly decreasing so that its inverse $f_{\sigma_t^2}^{(-1)}$ exists.

Proposition 5.6. *For $T \geq 0$ and $t \leq T$, the distribution function of $P(t,T)$ conditional on $\{\tau > t\}$ is*

$$\begin{aligned}
&\mathbf{P}(P(t,T) \leq x | \tau > t) \\
&\quad = \mathbb{E}\left[\mathrm{N}\left(\frac{-f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right) - \mathrm{N}\left(\frac{2b - f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right)\right] \Big/ \left[1 - 2\,\mathbb{E}\left(\mathrm{N}\left(\frac{b}{\sqrt{\Lambda_t}}\right)\right)\right], \\
&\qquad\qquad 0 \leq x < 1.
\end{aligned}$$

Proof. We have

$$\{P(t,T) \leq x\} = \left\{\mathbf{P}(\tau \in (t,T] | X_t, \sigma_t^2) \leq x, \tau > t\right\},$$

which establishes

$$\mathbf{P}(P(t,T) \leq x | \tau > t) = \frac{\mathbf{P}(P(t,T) \leq x)}{\mathbf{P}(\tau > t)}, \quad x < 1.$$

By the independence of the DDS-Brownian motion B and the pair σ_t^2, Λ_t, we obtain on

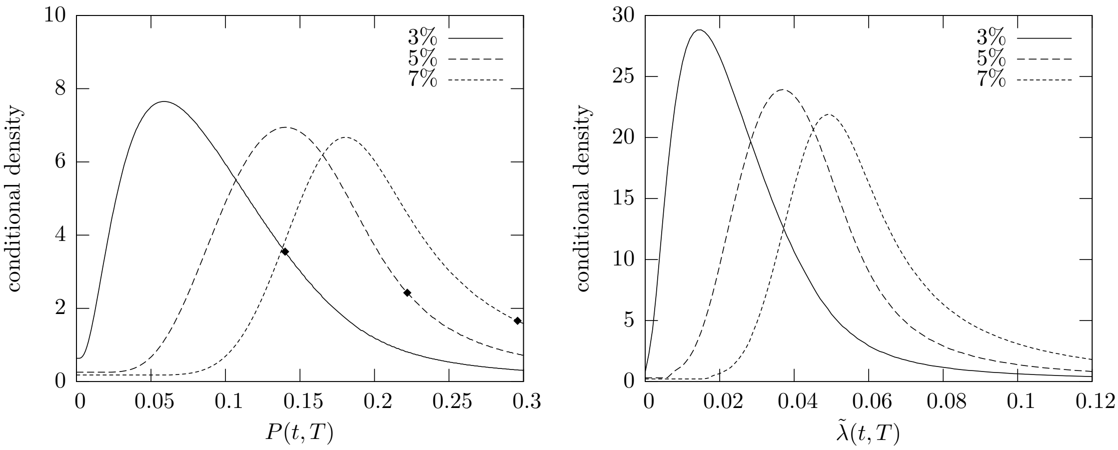

Figure 3: Distributions of $P(t,T)$ (left) and $\tilde{\lambda}(t,T)$ (right) conditional on $\{\tau > t\}$. The term hazard rate $\tilde{\lambda}(t,T)$ is an approximation of the credit spread, i.e., $s(t,T) \approx (1-R)\tilde{\lambda}(t,T)\cdot 10^4$ (in basis points). As in Figure 1, we have $t = 1, T = 5$ and initial hazard rates of 3%, 5%, 7%. The diamonds mark the initial 5-year default probability $\mathbf{P}(\tau \le T)$. The model parameters are mean reversion $a = 3$, recovery rate $R = 0.4$, barrier $b = -3$; σ^2 is driven by compound Poisson processes with jump intensity $\lambda = 2$ and jump size distributions $\{0.1, 20\}$ ($h = 0.03$), $\{0.2, 50\}$ ($h = 0.05$), $\{0.4, 100\}$ ($h = 0.07$) with probabilities $\{0.95, 0.05\}$. In each case, the the initial variance σ_0^2 and the function θ were chosen to fit the given term structure.

$\{\tau > t\}$,

$$\begin{aligned}
\mathbf{P}\left(P(t,T) \le x | \Lambda_t, \sigma_t^2\right) &= \mathbf{P}\left(f_{\sigma_t^2}(X_t) \le x, \min_{0<s\le t} X_s > b \Big| \Lambda_t, \sigma_t^2\right) \\
&= \mathbf{P}\left(f_{\sigma_t^2}(B_{\Lambda_t}) \le x, \min_{0<s\le \Lambda_t} B_s > b \Big| \Lambda_t, \sigma_t^2\right) \\
&= \mathbf{P}\left(B_{\Lambda_t} \ge f_{\sigma_t^2}^{(-1)}(x), \min_{0<s\le \Lambda_t} B_s > b \Big| \Lambda_t, \sigma_t^2\right) \\
&= \mathbf{P}\left(B_{\Lambda_t} \ge f_{\sigma_t^2}^{(-1)}(x) \Big| \Lambda_t, \sigma_t^2\right) - \mathbf{P}\left(B_{\Lambda_t} \ge f_{\sigma_t^2}^{(-1)}(x), \min_{0<s\le \Lambda_t} B_s \le b \Big| \Lambda_t, \sigma_t^2\right) \\
&= \mathrm{N}\left(\frac{-f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right) - \mathrm{N}\left(\frac{2b - f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right),
\end{aligned}$$

using the joint distribution of a Brownian motion and its running minimum as in Section 3.3. The claim now follows by taking expectation. □

Examples of the conditional distributions of $P(t,T)$ and the term hazard rate $\tilde{\lambda}(t,T)$, where the variance process follows an LOU process, are shown in Figure 3. The models are calibrated to the same term structures as the examples of the OS-model from Figure 1. Recall that the distributions in the OS-model suggested a very volatile movement of the credit quality process. By the introduction of jumps in the variance, the default speed in this example has been significantly reduced, while maintaining the default probabilities implied by the given term structures. Different distributions, and hence different dynamics, are obtained by changing the parameters of the LOU process. Further examples are discussed in the next section.

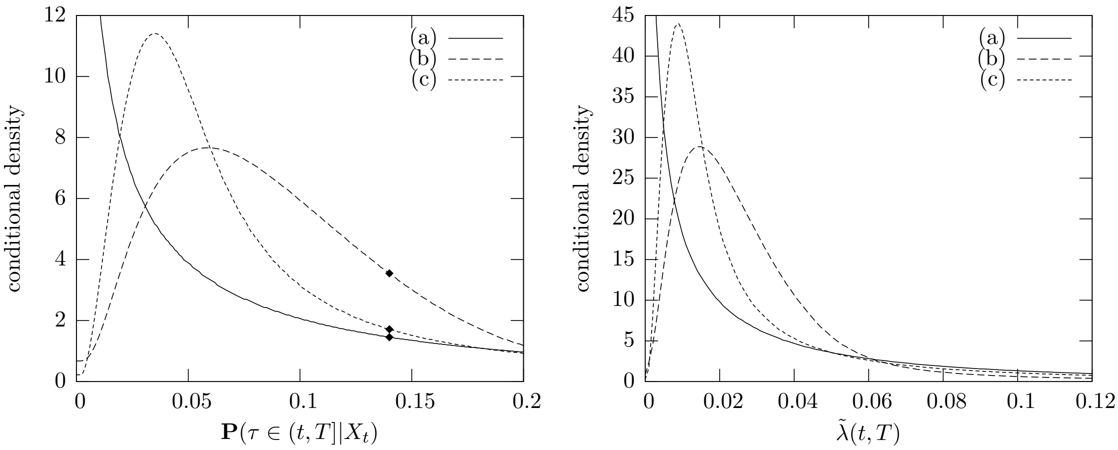

Figure 4: Distributions of $P(t,T)$ (left) and $\tilde{\lambda}(t,T)$ (right) conditional on $\{\tau > t\}$ for different dynamics. We choose $t = 1, T = 5$ and an initial hazard rate of 3%. The diamonds mark the initial 5-year default probability. The parameters are given in Table 1.

	(a)	**(b)**	**(c)**	**(d)**
		Parameters		
a	3	3	1	1
b	-3	-3	-3	-2
λ	0	2	1	0.0305
F		$\begin{bmatrix} 0.1 & (0.95) \\ 20 & (0.05) \end{bmatrix}$	$\begin{bmatrix} 0.1 & (0.95) \\ 10 & (0.05) \end{bmatrix}$	$\mathbf{1}_{\{(0,2.5\cdot 10^4]\}}$
σ_0^2	3.16	4.59	3.25	10^{-6}
θ	$\in [0.23, 1.32]$	$\in [-0.22, 0.04]$	$\in [-1.10, 0.37]$	0

Table 1: Parameters of dynamics examples. In each case, the jump process Z is a compound Poisson process with strictly positive jumps. The jump sizes are given by the first column and the corresponding probabilities in the second column of each matrix in the row of parameter F.

5.3 Examples

Calibration of default probabilities essentially means choosing the default barrier b and the parameters of the variance process σ^2 of Equation (12), i.e., mean reversion a, jump intensity λ, jump size distribution F, deterministic function θ and initial variance σ_0^2. The last two parameters, θ and σ_0^2, will be chosen such as to minimize the root mean square error between given default probabilities and model default probabilities. The other parameters determine the dynamics of the model. It is important to note, that the deterministic function θ influences the dynamics and that the parameters for the dynamics influence the calibration of θ. The overall calibration process is to assign parameters for the dynamics first and then to calibrate to the given spot curve. Again, details on calibration, calibration error, attainable term structures and examples are found in [Packham et al., 2009].

To illustrate the range of dynamics, Figure 4 shows the conditional distributions (conditional on no default) of the 5-year default probability and term hazard rate in one year for different parameter sets. The corresponding parameters are given in Table 1. In each case, the initial variance σ_0^2 and the function θ are calibrated to match default probabilities corresponding to an initial hazard rate of 3%.

Case (a) is a model with a deterministic time-change, and hence corresponds to the OS-model. Case (d) was chosen such that $\sigma_0^2 = 10^{-6}$ and $\theta \equiv 0$, so that the variance is very small until the first jump occurs. The jump size was chosen to be very large relative to the default barrier so that, heuristically, a single jump leads to default very quickly. Loosely speaking, case (d) can be considered an approximation of a reduced-form model with a deterministic and constant intensity: the credit quality process exhibits practically no movement, until the first jump occurs, which leads to default with a very high probability. This is also reflected in the jump intensity $\lambda = 0.0305$, which is approximately the initial hazard rate, and in $\mathbf{P}(\tau \in (1,5)|X_1, \sigma_1^2) \approx 1 - \mathbf{e}^{-0.03 \cdot 4} = 0.11308$ conditional on no default until time 1.[3] These two cases illustrate that the parameters can be classified into parameters that govern the jump part of the variance process, namely the jump intensity and jump size distribution, and parameters that control the continuous behaviour of the process in the sense that level of the function θ determines the minimum volatility of the credit quality process at any time. By calibration to a term structure, a low level of jump activity leads to a higher minimum volatility and vice versa.

The characteristics of cases (b) and (c) are "in-between" cases (a) and (d): in both cases, the variance process exhibits jumps. However, the jump dynamics are moderate enough for the level of the variance process induced by θ and σ_0^2, both of which are obtained by calibration to the given term structure, to be significantly above zero. In other words, the variance processes of both cases feature jumps and a significant minimum "default speed".

6 Information flow and the pricing filtration

So far, we have made no assumptions about the filtration $(\mathcal{F}_t)_{t\geq 0}$, other than that it is rich enough to carry $(\mathcal{F}_t)_{t\geq 0}$-adapted drivers of the credit quality process (X, σ^2). Following the principle of arbitrage pricing theory, the underlying filtration must be generated by the observable prices of traded assets, see e.g. Chapter 7 of [Hunt and Kennedy, 2004]. In general, a credit quality process (X, σ^2) is neither directly observable nor a traded asset.

Suppose now that we wish to price financial claims derived from credit spreads (or, equivalently, default, resp. survival probabilities). Application of a risk-neutral valuation formula with conditional default probabilities given by the model via Equation (7) (or credit spreads derived thereof) is justified only if $(\mathcal{F}_t)_{t\geq 0}$ is generated by some observable information and if X and σ^2 are $(\mathcal{F}_t)_{t\geq 0}$-adapted. Otherwise, valuation of assets requires that prices are computed using a different – possibly coarser – filtration. One may think of a coarser filtration as the inavailability in the market of complete information about a company's state.

Assuming independence of risk-free interest rates and the default indicator process, the filtration generated by risk-free zero-coupon bonds $(B(t,T))_{T\geq t}$ and conditional default probabilities $(P(t,T))_{T\geq t}$, $t \geq 0$, will be sufficient for this purpose; owing to the valuation formula for risky zero-coupon bonds, given by

$$\mathbb{E}\left(\mathbf{e}^{-\int_t^T r_s\,\mathrm{d}s}\,\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\right) = B(t,T)\,\mathbf{P}(\tau > T|\mathcal{F}_t), \quad T \geq t,$$

this filtration is equivalent to the filtration generated by risk-free and risky zero-coupon bonds (of all maturities).

The assumption that there is indeed a process that drives a company's credit quality via the information available about the company may be justified by the stylised facts recorded earlier, namely that the arrival of news about a company affects CDS spreads of

[3] If the initial hazard rate is not constant, then a calibration where the variance moves purely by jumps cannot be attained. This is due to the fact that the jump intensity of the variance's compound Poisson process is constant, whereas a non-constant, deterministic hazard rate requires the jump intensity to be non-constant and deterministic. The former can be incorporated by specifying the jump process as an additive process, which is a process with independent increments that is stochastically continuous, but whose increments are not necessarily stationary.

all maturities in a similar fashion. We shall assume that the credit quality of a firm is indeed driven by a process (X, σ^2) as in Proposition 4.8 with σ^2 an LOU process driven by a pure-jump subordinator (with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$). Furthermore, we assume that the parameters of the credit quality process are known, i.e., θ, σ_0^2, a, b, c, λ, F are $\mathcal{F}_0$-measurable. The following result establishes that, under these assumptions, the formula for conditional default probabilities is suitable for valuation.

Proposition 6.1. *Let (X, σ^2) be a credit quality process as in Proposition 4.8, with σ^2 an LOU process driven by a subordinator. Let $\mathcal{F}_t^P = \sigma((P(s,T)_{T>s}), 0 \leq s \leq t)$. Then, $\sigma(X_t, \sigma_t^2) \subseteq \mathcal{F}_t^P$. Moreover, $\mathbf{P}(\tau \leq T|\mathcal{F}_t^p) = \mathbf{P}(\tau \leq T|X_t, \sigma_t^2)$.*

Proof. As in the proof of Proposition 5.1, abbreviate

$$h(t,T) = \int_t^T \theta(u) \left(1 - \mathbf{e}^{-a(t-u)}\right) \mathrm{d}u$$

$$L_{t,T} = \int_t^T \left(1 - \mathbf{e}^{-a(T-u)}\right) \mathrm{d}Z_u,$$

so that, by Equation (13),

$$\Lambda_T - \Lambda_t = \left(1 - \mathbf{e}^{-a(T-t)}\right) \frac{\sigma_t^2}{a} + h(t,T) + \frac{L_{t,T}}{a}.$$

On $\{\tau > t\}$, by the Markov property of (X, σ^2) and by the independence of $L_{t,T}$ and (X_t, σ_t^2), a version of the conditional probability at t until T is given by $g_{t,T}(X_t, \sigma_t)$ with

$$g_{t,T}(x,y) := \mathbb{E}\left(2\mathrm{N}\left(\frac{b-x}{\sqrt{\left(1-\mathbf{e}^{-a(T-s)}\right)y^2/a + h(t,T) + L_{t,T}/a}}\right)\right).$$

Observe that for fixed x, $y \mapsto g_T(x,y)$ is invertible, and that for fixed y, $x \mapsto g_T(x,y)$ is invertible (both mappings being strictly monotone). By Proposition 5.1, $\Delta\sigma_t^2 > 0$ if and only if $\Delta P(t,T) > 0$, $T > t$. Moreover, the only "random" movements of σ^2 occur in jumps, any movement between jumps, conditional on the value attained by the last jump, is deterministic. Consequently, if $\Delta P(t,T) = 0$, $T > t$, and if the value of σ^2 at the last jump time is known, then σ_t^2 is deterministic, and X_t may be determined by inversion. On the other hand, if $\Delta P(t,T) > 0$, $T > t$, then by Equation (15),

$$\Delta P(t,T) = g_T(X_t, \sigma_t^2) - g_T(X_t, \sigma_{t-}^2), \quad t < T, \quad \mathbf{P}\text{–a.s.},$$

which can again be inferred by inversion (in particular, since $X_{t-} = X_t$). Hence, $(X_u, \sigma_u^2)_{u\leq t}$ is $\mathcal{F}_t^P$-measurable, and the first claim follows.

For the second claim observe that $\mathcal{F}_t^{W,Z} \subseteq \mathcal{F}_t^P \subseteq \mathcal{F}_t$, where $(\mathcal{F}_t^{W,Z})_{t\geq 0}$ denotes the filtration generated by W and Z. It follows that W is an $(\mathcal{F}_t^P)_{t\geq 0}$-Brownian motion and Z is a subordinator with respect to $(\mathcal{F}_t^P)_{t\geq 0}$, and the claim follows. □

Under some further assumptions on the deterministic function θ, it can be shown that $\sigma(X_t, \sigma_t^2) = \sigma(P(t,T)_{T>t})$, see Section 9.1 of [Packham, 2009].

The valuation of financial claims in the LOU variance model driven by a compound Poisson process is done by a combination of Monte Carlo simulation and numerical computation. Conditional on (X_t, σ_t^2), default probabilities $P(t,T) = \mathbf{P}(\tau \leq T|X_t, \sigma_t^2)$, $T > t$, can be computed numerically, using e.g. Panjer recursion to compute the distribution of the time-change. Monte Carlo simulation then reduces to simulating X_t and σ_t. The advantage of such an algorithm is that valuation of a product involving $P(t,T)$ or $s(t,T)$ requires simulation only until time t instead of T. For example, valuation of a default swaption requires simulation until expiry of the option instead of simulation until maturity of the underlying CDS. See [Packham et al., 2009] for details and the full algorithm.

A The term hazard rate

We sometimes consider the *term hazard rate* defined by

$$\tilde{\lambda}(t,T) = -\frac{\ln(1-P(t,T))}{T-t}, \quad T > t, \quad P(t,T) < 1, \tag{19}$$

as a proxy for the credit spread $s(t,T)/(1-R)$. The use of the term hazard rate is motivated by the fact that $\tilde{\lambda}(t,T)$ is a function of $P(t,T)$ instead of $(P(t,T))_{T\geq t}$ as is the case for the spread $s(t,T)$. That it may be considered a proxy for the credit spread is explained as follows: If the default time admits a (conditional) density, then the hazard rate at time t is the mapping $T \mapsto \lambda(t,T)$ defined by

$$\lambda(t,T) = -\frac{\mathrm{d}}{\mathrm{d}T}\ln(1-P(t,T)), \quad T \geq t.$$

It follows easily that

$$1 - P(t,T) = \mathbf{e}^{-\int_t^T \lambda(t,u)\,\mathrm{d}u},$$

which, together with Equation (1) yields the relationship

$$\frac{s(t,T)}{1-R}\int_t^T \mathbf{e}^{-r(u-t)}(1-P(t,u))\,\mathrm{d}u = \int_t^T \lambda(t,u)\,\mathbf{e}^{-r(u-t)}(1-P(t,u))\,\mathrm{d}u.$$

An approximation of the right-hand side is

$$\tilde{\lambda}(t,T)\int_t^T \mathbf{e}^{-r(u-t)}(1-P(t,u))\,\mathrm{d}u,$$

with

$$\tilde{\lambda}(t,T) = \frac{\int_t^T \lambda(t,u)\,\mathrm{d}u}{T-t},$$

which yields the well-known credit triangle

$$\frac{s(t,T)}{1-R} \approx \tilde{\lambda}(t,T).$$

B Ocone martingales

The famous Theorem of Dambis, Dubins-Schwarz (DDS-Theorem) (see e.g. [Karatzas and Shreve, 1998, Chapter 3.4.B]) states that a continuous local martingale has a representation as a time-changed Brownian motion. *Ocone martingales* are those continuous local martingales whose DDS-Brownian motion and associated time-change are independent, see [Ocone, 1993], [Dubins et al., 1993], [Vostrikova and Yor, 2000]. We study the relationship between stochastic integrals (with respect to Brownian motion) and Ocone martingales. The result is applied in Section 4.2 to derive conditional default probabilities in the credit quality process model.

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ that satisfies the usual hypotheses. For a continuous local martingale M, let us call the Brownian motion in its representation as a time-changed Brownian motion the *DDS-Brownian motion of* M, and let us denote denote the quadratic variation of M by $[M,M]$.

Definition B.1. A continuous local martingale $M = (M_t)_{t\geq 0}$, $M_0 = 0$, is an *Ocone martingale*, if its DDS-Brownian motion is independent of $[M,M]$.

We are interested in identifying Ocone martingales that are defined as stochastic integrals with respect to a Brownian motion. [Vostrikova and Yor, 2000] provide a characterisation for adapted, continuous integrands. We shall allow adapted càdlàg processes as integrands.

Proposition B.2. *Let $W = (W_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and let $\sigma = (\sigma_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted, strictly positive càdlàg process such that $\mathbf{P}(\int_0^t \sigma_s^2 \,\mathrm{d}s < \infty) = 1$, $t \geq 0$, and $\mathbf{P}(\lim_{t\to\infty} \int_0^t \sigma_s^2 \,\mathrm{d}s = \infty) = 1$. Let $M = (M_t)_{t\geq 0}$ be the continuous local martingale given by*

$$M_t = \int_0^t \sigma_s \,\mathrm{d}W_s, \quad t \geq 0.$$

Then, W and σ are independent if and only if M is an Ocone martingale.

Observe that $[M,M] = \int_0^\cdot \sigma_s^2 \,\mathrm{d}s$. For the proof, we need the following Lemma.

Lemma B.3. *Let W, σ, M be as in Proposition B.2. Then, $\sigma([M,M]_t, 0 \leq t < \infty) = \sigma(\sigma_t^2, 0 \leq t < \infty)$.*

Proof. Clearly, $\sigma([M,M]_t, 0 \leq t < \infty) \subseteq \sigma(\sigma_t^2, 0 \leq t < \infty)$. For the converse statement, observe first that $[M,M]$ is differentiable a.e. (with respect to Lebesgue measure), and for every such t, $[M,M]'_t = \sigma_t^2$. Conversely, at any point t where σ^2 is continuous, $[M,M]$ is differentiable, for suppose that σ^2 is continuous at t, but

$$\lim_{h\downarrow 0} \frac{[M,M]_{t+h} - [M,M]_t}{h} \neq \lim_{h\uparrow 0} \frac{[M,M]_{t+h} - [M,M]_t}{h};$$

this is a contradiction to $[M,M]' = \sigma^2$ a.e. and the continuity of σ^2 at t. Now, let t be a point where $[M,M]$ is not differentiable; i.e., σ^2 has a jump at t. From the right-continuity of σ^2, it follows that there exists $h > 0$ such that $[M,M]$ is differentiable on $(t, t+h)$. By the Mean Value Theorem, there exists $s_h \in (t, t+h)$ such that

$$\frac{[M,M]_{t+h} - [M,M]_t}{h} = \sigma_{s_h}^2.$$

Moreover, by the right-continuity of σ^2, taking the right limit,

$$\lim_{h\downarrow 0} \frac{[M,M]_{t+h} - [M,M]_t}{h} = \lim_{h\downarrow 0} \sigma_{s_h}^2 = \sigma_t^2,$$

and $\sigma(\sigma_t^2, 0 \leq t < \infty) \subseteq \sigma([M,M]_t, 0 \leq t < \infty)$ follows (see e.g. Lemma 1.13 of [Kallenberg, 2001]). □

Proof of Proposition B.2. Define the family of stopping times $(\tau_t)_{t\geq 0}$ by

$$\tau_t = \inf\left\{s \geq 0 : \int_0^s \sigma_u^2 \,\mathrm{d}u > t\right\}, \quad t \geq 0.$$

Then $B = (B_t)_{t\geq 0}$, given by $B_t = M_{\tau_t} = \int_0^{\tau_t} \sigma_s \,\mathrm{d}W_s$, $t \geq 0$, is the DDS-Brownian motion of M. It is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-Brownian motion.

The "only if" part: Let C be the space of real-valued continuous functions on $\mathbb{R}_+$ and let D be the space of real-valued càdlàg functions on $\mathbb{R}_+$. Denote by $(C, \mathcal{B}(C))$, resp. $(D, \mathcal{B}(D))$, the measurable space of real-valued continuous, resp. càdlàg , functions on $\mathbb{R}_+$ endowed with the σ-algebra generated by the finite-dimensional cylinder sets of C, resp. D.[4] For every $\Gamma \in \mathcal{B}(C)$ and $\Delta \in \mathcal{B}(D)$ we show that

$$\mathbf{P}(B \in \Gamma, \sigma \in \Delta) = \mathbf{P}(B \in \Gamma)\,\mathbf{P}(\sigma \in \Delta). \tag{20}$$

[4] Under a suitable metric on C, resp. D, the σ-algebra $\mathcal{B}(C)$, resp. $\mathcal{B}(D)$, corresponds to the σ-algebra generated by the open sets (with respect to the metric) of C, resp. D, see e.g. [Shiryaev, 1996, Section II.§2] or [Karatzas and Shreve, 1998, Sections 2.4 and 6.2].

That B is independent of $[M, M]$ and hence an Ocone martingale then follows by Lemma B.3.

It is straightforward that Equation (20) holds for sets Δ with $\mathbf{P}(\sigma \in \Delta) \in \{0, 1\}$. Choose Δ such that $\mathbf{P}(\sigma \in \Delta) \in (0, 1)$, and denote by $\mathscr{D}$ the σ-algebra generated by $\{\sigma \in \Delta\}$. By properties of conditional expectation,

$$\mathbf{P}(B \in \Gamma, \sigma \in \Delta) = \mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\mathbf{P}(B \in \Gamma|\mathscr{D})). \tag{21}$$

Writing $D_1 = \{\sigma \in \Delta\}$ and $D_2 = \{\sigma \notin \Delta\}$, it is easy to check that a version of the conditional probability of $A \in \mathcal{F}$ with respect to $\mathscr{D}$ is given by

$$\mathbf{P}(A|\mathscr{D})(\omega) = \sum_{i=1,2} \mathbf{P}(A \cap D_i)/\mathbf{P}(D_i)\mathbf{1}_{\{D_i\}}(\omega), \quad \omega \in \Omega.$$

Fix this version of the conditional probability. For every $\omega \in \Omega$, $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ is a probability measure (and thus it is a variant of the regular conditional probability with respect to $\mathscr{D}$. Moreover, $\mathbf{P}(\cdot|\mathscr{D})(\omega) \ll \mathbf{P}$, i.e., $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ is absolutely continuous with respect to $\mathbf{P}$. It follows, e.g. by Theorem 14 of [Protter, 2005, Section II.5], that $\int_0^\cdot \sigma_s \,\mathrm{d}W_s$ computed under the law $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ and M are $\mathbf{P}(\cdot|\mathscr{D})(\omega)$-indistinguishable.

By independence of W and $\mathscr{D}$ it follows that W is a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, and hence M is a continuous local martingale under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$. The quadratic variation, as a limit in probability, is invariant to absolutely continuous changes in measure. Hence, by the Lévy-characterisation of Brownian motion, B is an $(\mathcal{F}_{\tau_t})$-Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, in other words $\mathbf{P}(B \in \cdot|\mathscr{D}) = \mathbf{P}(B \in \cdot)$ is the Wiener measure on $(C, \mathcal{B}(C))$. Finally, inserting into Equation (21) yields

$$\mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\,\mathbf{P}(B \in \Gamma|\mathscr{D})) = \mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\,\mathbf{P}(B \in \Gamma)) = \mathbf{P}(B \in \Gamma)\,\mathbf{P}(\sigma \in \Delta). \tag{22}$$

The "if" part: Now suppose that M is an Ocone martingale, i.e., B and $[M, M]$ are independent. We have

$$W_t = \int_0^t \frac{1}{\sigma_s}\,\mathrm{d}M_s = \int_0^{[M,M]_t} \frac{1}{\sigma_{\tau_s}}\,\mathrm{d}B_s, \quad \mathbf{P}\text{–a.s.},$$

where the last part follows from [Karatzas and Shreve, 1998, Proposition 3.4.8]. Now it can be shown that, for $\Gamma, \Delta \in \mathcal{B}(C)$),

$$\mathbf{P}(W \in \Gamma, [M, M] \in \Delta) = \mathbf{P}(W \in \Gamma)\,\mathbf{P}([M, M] \in \Delta)$$

using the same technique as in the "only if" part of the proof. That W and σ are independent then follows from Lemma B.3. □

Remark B.4. For simplicity, we have required that the integrand σ be strictly positive. It is not hard to extend Proposition B.2 to integrands that are $\mathbf{P}$–a.s. nonzero, see e.g. [Vostrikova and Yor, 2000, Theorem 3].

In Proposition B.2 we started out with a continuous local martingale defined as a stochastic integral with respect to a Brownian motion. We may also consider the converse, where we define a continuous local martingale M as a Brownian motion and an independent time-change. Then, a sufficient condition for M to have a representation as a stochastic integral with respect to a Brownian motion is that $[M, M]$ be an absolutely continuous function of t for $\mathbf{P}$-almost every $\omega \in \Omega$, cf. [Karatzas and Shreve, 1998, Theorem 3.4.2].

We conclude this section by establishing some properties related to Proposition B.2 that are used in Section 4.2. For a stochastic process X, we denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the filtration generated by X.

Corollary B.5. *Let $M = \int_0^{\cdot} \sigma_s \, \mathrm{d}W_s$ be as in Proposition B.2, and let B the DDS-Brownian motion of M. Define the family of stopping times $(\tau_t)_{t \geq 0}$ by $\tau_t = \inf\left\{s \geq 0 : \int_0^s \sigma_u^2 \, \mathrm{d}u > t\right\}$, $t \geq 0$. Furthermore, let S be an $\mathbf{P}$–a.s. finite $(\mathcal{F}_{\tau_t})$-stopping time and define $\tilde{B} = (\tilde{B}_u)_{u \geq 0}$, with $\tilde{B}_u := B_{S+u} - B_S$. If M is an Ocone martingale, then*

(i) $\tilde{B}$ is an $(\mathcal{F}_t^{\tilde{B}})_{t \geq 0}$-Brownian motion independent of σ;

(ii) $(\mathcal{F}_t^{\tilde{B}})_{t \geq 0}$ and $\mathcal{F}_{\tau_S}$ are conditionally independent given $\mathcal{F}_\infty^\sigma$, the σ-algebra generated by σ;

(iii) $\tilde{B}$ is independent of $\mathcal{F}_{\tau_S} \vee \mathcal{F}_\infty^\sigma$, the smallest σ-algebra containing $\mathcal{F}_{\tau_S}$ and $\mathcal{F}_\infty^\sigma$.

Proof. (i) By the properties of Brownian motion, $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_{\tau_S}$, cf. [Karatzas and Shreve, 1998, Theorem 2.6.16]. In the notation of the previous proof, let $\Delta \in \mathcal{B}(D)$ and let $\mathscr{D}$ be the σ-algebra generated by $\{\sigma \in \Delta\}$. Since B is a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, for $\mathbf{P}$-almost all $\omega \in \Omega$, so is $\tilde{B}$, and the first claim follows in analogy to Equation (22).

(ii) The second claim is a generalisation of the first claim. Since B is independent of σ, we have $\mathbf{P}(B \in \Gamma|\mathcal{F}_\infty^\sigma) = \mathbf{P}(B \in \Gamma)$ $\mathbf{P}$–a.s., $\Gamma \in \mathcal{B}(C)$. Moreover, there exists a version $\mathbf{Q}$ of $\mathbf{P}(B \in \cdot|\mathcal{F}_\infty^\sigma)$ that is a regular conditional distribution of B with respect to $\mathcal{F}_\infty^\sigma$. It follows that B is a Brownian motion under $\mathbf{Q}$ and hence $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_{\tau_S}$ under $\mathbf{Q}$, which is the required result.

(iii) Since $\mathcal{F}_\infty^{\tilde{B}}$ and $\mathcal{F}_{\tau_S}$ are conditionally independent given $\mathcal{F}_\infty^\sigma$, $\mathbf{P}$–a.s. (see Proposition 6.6 of [Kallenberg, 2001] for the first statement),

$$\mathbf{P}(\tilde{B} \in \Gamma|\mathcal{F}_\infty^\sigma \vee \mathcal{F}_{\tau_S}) = \mathbf{P}(\tilde{B} \in \Gamma|\mathcal{F}_\infty^\sigma) = \mathbf{P}(\tilde{B} \in \Gamma), \quad \Gamma \in \mathcal{B}(C).$$

□

References

Ole E. Barndorff-Nielsen and Neil Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial econometrics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63:167–241, 2001.

Martin Baxter. Gamma process dynamic modelling of credit. *RISK*, October:98–101, October 2007.

Jean Bertoin. *Lévy Processes*. Cambridge University Press, 1998.

Fischer Black and John C. Cox. Valuing corporate securities: some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367, May 1976.

Jessica Cariboni and Wim Schoutens. Pricing credit default swaps under Lévy models. *Journal of Computational Finance*, 10(4):71–91, 2007.

Rama Cont and Peter Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, 2004.

Qiang Dai and Kenneth Singleton. Term structure dynamics in theory and reality. *The Review of Financial Studies*, 16(3):631–678, 2003.

S.R. Das. The surprise element: jumps in interest rates. *Journal of Econometrics*, 106(1):27–65, 2002.

L. E. Dubins, M. Émery, and M. Yor. On the Lévy Transformation of Brownian Motion and Continuous Martingales. *In: Séminaire de Probabilités XXVII, Springer*, pages 122–132, 1993.

Darrell Duffie and David Lando. Term structure of credit spreads with incomplete accounting information. *Econometrica*, 69:633–664, 2001.

Darrell Duffie and Kenneth J Singleton. Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12(4):687–720, 1999.

Darrell Duffie and Kenneth J. Singleton. *Credit Risk: Pricing, Measurement, and Management*. Princeton University Press, 2003.

J. Helwege and C.M. Turner. The slope of the credit yield curve for speculative-grade issuers. *The Journal of Finance*, 54(5):1869–1884, 1999.

P. J. Hunt and J. E. Kennedy. *Financial Derivatives in Theory and Practice*. Wiley, 2004.

Robert A. Jarrow and Stuart M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *The Journal of Finance*, 50(1):53–85, 1995.

M. S. Johannes. A nonparametric view of the role of jumps to interest rates. Working paper, Columbia Business School, 2000.

Olav Kallenberg. *Foundations of Modern Probability*. Springer, 2nd edition, 2001.

Stefanie Kammer. *A general first-passage time model for multivariate credit-spreads and a note on barrier option pricing*. PhD thesis, Justus-Liebig-Universität Giessen, September 2007.

Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 2nd edition, 1998.

Rüdiger Kiesel and Matthias Scherer. Dynamic credit portfolio modelling in structural models with jumps. Working paper, December 2007.

David Lando. On Cox processes and credit risky securities. *Review of Derivatives Research*, 2: 99–120, 1998.

David Lando. *Credit Risk Modeling*. Princeton University Press, 2004.

Francis A. Longstaff and Eduardo S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, July 1995.

Robert C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, May 1974.

Ragnar Norberg. Vasicek beyond the normal. *Mathematical Finance*, 14(4):585–604, 2004.

Daniel L. Ocone. A symmetry characterization of conditionally independent increment martingales. *In: Barcelona Seminar on Stochastic Analysis (St. Feliu de Guíxols, 1991), Birkhäuser*, 32:147–167, 1993.

Ludger Overbeck and Wolfgang M. Schmidt. Modeling default dependence with threshold models. *Journal of Derivatives*, 12(4):10–19, Summer 2005.

Natalie Packham. *Credit dynamics in a first-passage time model with jumps and Latin hypercube sampling with dependence*. PhD thesis, Frankfurt School of Finance & Management, 2009.

Natalie Packham, Lutz Schlögl, and Wolfgang M. Schmidt. Credit gap risk in a first-passage time model with jumps. Working paper, 2009.

Philip E. Protter. *Stochastic Integration and Differential Equations*. Springer, 2nd edition, 2005.

K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

Paul Schneider, Leopold Sögner, and Tanja Veza. The economic role of jumps and recovery rates in the market for corporate default risk. Working Paper (Journal of Financial and Quantitative Analysis; forthcoming), November 2007.

Wim Schoutens. *Lévy Processes in Finance*. John Wiley & Sons, 2003.

A. N. Shiryaev. *Probability*. Springer, 2nd edition, 1996.

George Tauchen and Hao Zhou. Realized jumps on financial markets and predicting credit spreads. FEDS Working Paper No. 2006-35 (Journal of Econometrics, forthcoming), August 2006.

L. Vostrikova and M. Yor. Some invariance properties (of the laws) of Ocone's martingales. *In: Séminaire de Probabilités XXXIV, Springer*, pages 417–431, 2000. [One page of this article is misplaced: Page 421 should be read after page 425, not page 420.].

Benjamin Yibin Zhang, Hao Zhou, and Haibin Zhu. Explaining credit default swap spreads with the equity volatility and jump risks of individual firms. *Review of Financial Studies*, 2009. doi: 10.1093/rfs/hhp004.

Chunsheng Zhou. The term structure of credit spreads with jump risk. *Journal of Banking and Finance*, 25:2015–2040, 2001.

FRANKFURT SCHOOL / HFB – WORKING PAPER SERIES

No.	Author/Title	Year
129.	Hankir, Yassin / Rauch, Christian / Umber, Marc It's The Market Power, Stupid! – Stock Return Patterns in International Bank M&A	2009
128.	Herrmann-Pillath, Carsten Outline of a Darwinian Theory of Money	2009
127.	Cremers, Heinz / Walzner, Jens Modellierung des Kreditrisikos im Portfoliofall	2009
126.	Cremers, Heinz / Walzner, Jens Modellierung des Kreditrisikos im Einwertpapierfall	2009
125.	Heidorn, Thomas / Schmaltz, Christian Interne Transferpreise für Liquidität	2009
124.	Bannier, Christina E. / Hirsch, Christian The economic function of credit rating agencies - What does the watchlist tell us?	2009
123.	Herrmann-Pillath, Carsten A Neurolinguistic Approach to Performativity in Economics	2009
122.	Winkler, Adalbert / Vogel, Ursula Finanzierungsstrukturen und makroökonomische Stabilität in den Ländern Südosteuropas, der Türkei und in den GUS-Staaten	2009
121.	Heidorn, Thomas / Rupprecht, Stephan Einführung in das Kapitalstrukturmanagement bei Banken	2009
120.	Rossbach, Peter Die Rolle des Internets als Informationsbeschaffungsmedium in Banken	2009
119.	Herrmann-Pillath, Carsten Diversity Management und diversi-tätsbasiertes Controlling: Von der „Diversity Scorecard“ zur „Open Balanced Scorecard	2009
118.	Hölscher, Luise / Clasen, Sven Erfolgsfaktoren von Private Equity Fonds	2009
117.	Bannier, Christina E. Is there a hold-up benefit in heterogeneous multiple bank financing?	2009
116.	Roßbach, Peter / Gießamer, Dirk Ein eLearning-System zur Unterstützung der Wissensvermittlung von Web-Entwicklern in Sicherheitsthemen	2009
115.	Herrmann-Pillath, Carsten Kulturelle Hybridisierung und Wirtschaftstransformation in China	2009
114.	Schalast, Christoph: Staatsfonds – „neue“ Akteure an den Finanzmärkten?	2009
113.	Schalast, Christoph / Alram, Johannes Konstruktion einer Anleihe mit hypothekarischer Besicherung	2009
112.	Schalast, Christoph / Bolder, Markus / Radünz, Claus / Siepmann, Stephanie / Weber, Thorsten Transaktionen und Servicing in der Finanzkrise: Berichte und Referate des Frankfurt School NPL Forums 2008	2009
111.	Werner, Karl / Moormann, Jürgen Efficiency and Profitability of European Banks – How Important Is Operational Efficiency?	2009
110.	Herrmann-Pillath, Carsten Moralische Gefühle als Grundlage einer wohlstandschaffenden Wettbewerbsordnung: Ein neuer Ansatz zur erforschung von Sozialkapital und seine Anwendung auf China	2009
109.	Heidorn, Thomas / Kaiser, Dieter G. / Roder, Christoph Empirische Analyse der Drawdowns von Dach-Hedgefonds	2009
108.	Herrmann-Pillath, Carsten Neuroeconomics, Naturalism and Language	2008
107.	Schalast, Christoph / Benita, Barten Private Equity und Familienunternehmen – eine Untersuchung unter besonderer Berücksichtigung deutscher Maschinen- und Anlagenbauunternehmen	2008
106.	Bannier, Christina E. / Grote, Michael H. Equity Gap? – Which Equity Gap? On the Financing Structure of Germany's Mittelstand	2008
105.	Herrmann-Pillath, Carsten The Naturalistic Turn in Economics: Implications for the Theory of Finance	2008

104. Schalast, Christoph (Hrgs.) / Schanz, Kay-Michael / Scholl, Wolfgang
Aktionärsschutz in der AG falsch verstanden? Die Leica-Entscheidung des LG Frankfurt am Main 2008

103. Bannier, Christina E./ Müsch, Stefan
Die Auswirkungen der Subprime-Krise auf den deutschen LBO-Markt für Small- und MidCaps 2008

102. Cremers, Heinz / Vetter, Michael
Das IRB-Modell des Kreditrisikos im Vergleich zum Modell einer logarithmisch normalverteilten Verlustfunktion 2008

101. Heidorn, Thomas / Pleißner, Mathias
Determinanten Europäischer CMBS Spreads. Ein empirisches Modell zur Bestimmung der Risikoaufschläge von Commercial Mortgage-Backed Securities (CMBS) 2008

100. Schalast, Christoph (Hrsg.) / Schanz, Kay-Michael
Schaeffler KG/Continental AG im Lichte der CSX Corp.-Entscheidung des US District Court for the Southern District of New York 2008

99. Hölscher, Luise / Haug, Michael / Schweinberger, Andreas
Analyse von Steueramnestiedaten 2008

98. Heimer, Thomas / Arend, Sebastian
The Genesis of the Black-Scholes Option Pricing Formula 2008

97. Heimer, Thomas / Hölscher, Luise / Werner, Matthias Ralf
Access to Finance and Venture Capital for Industrial SMEs 2008

96. Böttger, Marc / Guthoff, Anja / Heidorn, Thomas
Loss Given Default Modelle zur Schätzung von Recovery Rates 2008

95. Almer, Thomas / Heidorn, Thomas / Schmaltz, Christian
The Dynamics of Short- and Long-Term CDS-spreads of Banks 2008

94. Barthel, Erich / Wollersheim, Jutta
Kulturunterschiede bei Mergers & Acquisitions: Entwicklung eines Konzeptes zur Durchführung einer Cultural Due Diligence 2008

93. Heidorn, Thomas / Kunze, Wolfgang / Schmaltz, Christian
Liquiditätsmodellierung von Kreditzusagen (Term Facilities and Revolver) 2008

92. Burger, Andreas
Produktivität und Effizienz in Banken – Terminologie, Methoden und Status quo 2008

91. Löchel, Horst / Pecher, Florian
The Strategic Value of Investments in Chinese Banks by Foreign Financial Insitutions 2008

90. Schalast, Christoph / Morgenschweis, Bernd / Sprengetter, Hans Otto / Ockens, Klaas / Stachuletz, Rainer / Safran, Robert
Der deutsche NPL Markt 2007: Aktuelle Entwicklungen, Verkauf und Bewertung – Berichte und Referate des NPL Forums 2007 2008

89. Schalast, Christoph / Stralkowski, Ingo
10 Jahre deutsche Buyouts 2008

88. Bannier, Christina E./ Hirsch, Christian
The Economics of Rating Watchlists: Evidence from Rating Changes 2007

87. Demidova-Menzel, Nadeshda / Heidorn, Thomas
Gold in the Investment Portfolio 2007

86. Hölscher, Luise / Rosenthal, Johannes
Leistungsmessung der Internen Revision 2007

85. Bannier, Christina / Hänsel, Dennis
Determinants of banks' engagement in loan securitization 2007

84. Bannier, Christina
“Smoothing“ versus “Timeliness“ - Wann sind stabile Ratings optimal und welche Anforderungen sind an optimale Berichtsregeln zu stellen? 2007

83. Bannier, Christina E.
Heterogeneous Multiple Bank Financing: Does it Reduce Inefficient Credit-Renegotiation Incidences? 2007

82. Cremers, Heinz / Löhr, Andreas
Deskription und Bewertung strukturierter Produkte unter besonderer Berücksichtigung verschiedener Marktszenarien 2007

81. Demidova-Menzel, Nadeshda / Heidorn, Thomas
Commodities in Asset Management 2007

80. Cremers, Heinz / Walzner, Jens
Risikosteuerung mit Kreditderivaten unter besonderer Berücksichtigung von Credit Default Swaps 2007

79. Cremers, Heinz / Traughber, Patrick
Handlungsalternativen einer Genossenschaftsbank im Investmentprozess unter Berücksichtigung der Risikotragfähigkeit 2007

78. Gerdesmeier, Dieter / Roffia, Barbara
Monetary Analysis: A VAR Perspective 2007

77. Heidorn, Thomas / Kaiser, Dieter G. / Muschiol, Andrea
Portfoliooptimierung mit Hedgefonds unter Berücksichtigung höherer Momente der Verteilung 2007

76. Jobe, Clemens J. / Ockens, Klaas / Safran, Robert / Schalast, Christoph
Work-Out und Servicing von notleidenden Krediten – Berichte und Referate des HfB-NPL Servicing Forums 2006 2006

75. Abrar, Kamyar / Schalast, Christoph
Fusionskontrolle in dynamischen Netzsektoren am Beispiel des Breitbandkabelsektors 2006

74. Schalast, Christoph / Schanz, Kay-Michael
Wertpapierprospekte: Markteinführungspublizität nach EU-Prospektverordnung und Wertpapierprospektgesetz 2005 2006

73. Dickler, Robert A. / Schalast, Christoph
Distressed Debt in Germany: What´s Next? Possible Innovative Exit Strategies 2006

72. Belke, Ansgar / Polleit, Thorsten
How the ECB and the US Fed set interest rates 2006

71. Heidorn, Thomas / Hoppe, Christian / Kaiser, Dieter G.
Heterogenität von Hedgefondsindizes 2006

70. Baumann, Stefan / Löchel, Horst
The Endogeneity Approach of the Theory of Optimum Currency Areas - What does it mean for ASEAN + 3? 2006

69. Heidorn, Thomas / Trautmann, Alexandra
Niederschlagsderivate 2005

68. Heidorn, Thomas / Hoppe, Christian / Kaiser, Dieter G.
Möglichkeiten der Strukturierung von Hedgefondsportfolios 2005

67. Belke, Ansgar / Polleit, Thorsten
(How) Do Stock Market Returns React to Monetary Policy ? An ARDL Cointegration Analysis for Germany 2005

66. Daynes, Christian / Schalast, Christoph
Aktuelle Rechtsfragen des Bank- und Kapitalmarktsrechts II: Distressed Debt - Investing in Deutschland 2005

65. Gerdesmeier, Dieter / Polleit, Thorsten
Measures of excess liquidity 2005

64. Becker, Gernot M. / Harding, Perham / Hölscher, Luise
Financing the Embedded Value of Life Insurance Portfolios 2005

63. Schalast, Christoph
Modernisierung der Wasserwirtschaft im Spannungsfeld von Umweltschutz und Wettbewerb – Braucht Deutschland eine Rechtsgrundlage für die Vergabe von Wasserversorgungskonzessionen? – 2005

62. Bayer, Marcus / Cremers, Heinz / Kluß, Norbert
Wertsicherungsstrategien für das Asset Management 2005

61. Löchel, Horst / Polleit, Thorsten
A case for money in the ECB monetary policy strategy 2005

60. Richard, Jörg / Schalast, Christoph / Schanz, Kay-Michael
Unternehmen im Prime Standard - „Staying Public“ oder „Going Private“? - Nutzenanalyse der Börsennotiz - 2004

59. Heun, Michael / Schlink, Torsten
Early Warning Systems of Financial Crises - Implementation of a currency crisis model for Uganda 2004

58. Heimer, Thomas / Köhler, Thomas
Auswirkungen des Basel II Akkords auf österreichische KMU 2004

57. Heidorn, Thomas / Meyer, Bernd / Pietrowiak, Alexander
Performanceeffekte nach Directors´Dealings in Deutschland, Italien und den Niederlanden 2004

56. Gerdesmeier, Dieter / Roffia, Barbara
The Relevance of real-time data in estimating reaction functions for the euro area 2004

55. Barthel, Erich / Gierig, Rauno / Kühn, Ilmhart-Wolfram
Unterschiedliche Ansätze zur Messung des Humankapitals 2004

54. Anders, Dietmar / Binder, Andreas / Hesdahl, Ralf / Schalast, Christoph / Thöne, Thomas
Aktuelle Rechtsfragen des Bank- und Kapitalmarktrechts I :
Non-Performing-Loans / Faule Kredite - Handel, Work-Out, Outsourcing und Securitisation 2004

53. Polleit, Thorsten
The Slowdown in German Bank Lending – Revisited 2004

52. Heidorn, Thomas / Siragusano, Tindaro
Die Anwendbarkeit der Behavioral Finance im Devisenmarkt 2004

51. Schütze, Daniel / Schalast, Christoph (Hrsg.)
Wider die Verschleuderung von Unternehmen durch Pfandversteigerung 2004

50. Gerhold, Mirko / Heidorn, Thomas
Investitionen und Emissionen von Convertible Bonds (Wandelanleihen) 2004

49. Chevalier, Pierre / Heidorn, Thomas / Krieger, Christian
Temperaturderivate zur strategischen Absicherung von Beschaffungs- und Absatzrisiken 2003

48. Becker, Gernot M. / Seeger, Norbert
Internationale Cash Flow-Rechnungen aus Eigner- und Gläubigersicht 2003

47. Boenkost, Wolfram / Schmidt, Wolfgang M.
Notes on convexity and quanto adjustments for interest rates and related options 2003

46. Hess, Dieter
Determinants of the relative price impact of unanticipated Information in U.S. macroeconomic releases 2003

45. Cremers, Heinz / Kluß, Norbert / König, Markus
Incentive Fees. Erfolgsabhängige Vergütungsmodelle deutscher Publikumsfonds 2003

44. Heidorn, Thomas / König, Lars
Investitionen in Collateralized Debt Obligations 2003

43. Kahlert, Holger / Seeger, Norbert
Bilanzierung von Unternehmenszusammenschlüssen nach US-GAAP 2003

42. Beiträge von Studierenden des Studiengangs BBA 012 unter Begleitung von Prof. Dr. Norbert Seeger
Rechnungslegung im Umbruch - HGB-Bilanzierung im Wettbewerb mit den internationalen Standards nach IAS und US-GAAP 2003

41. Overbeck, Ludger / Schmidt, Wolfgang
Modeling Default Dependence with Threshold Models 2003

40. Balthasar, Daniel / Cremers, Heinz / Schmidt, Michael
Portfoliooptimierung mit Hedge Fonds unter besonderer Berücksichtigung der Risikokomponente 2002

39. Heidorn, Thomas / Kantwill, Jens
Eine empirische Analyse der Spreadunterschiede von Festsatzanleihen zu Floatern im Euroraum und deren Zusammenhang zum Preis eines Credit Default Swaps 2002

38. Böttcher, Henner / Seeger, Norbert
Bilanzierung von Finanzderivaten nach HGB, EstG, IAS und US-GAAP 2003

37. Moormann, Jürgen
Terminologie und Glossar der Bankinformatik 2002

36. Heidorn, Thomas
Bewertung von Kreditprodukten und Credit Default Swaps 2001

35. Heidorn, Thomas / Weier, Sven
Einführung in die fundamentale Aktienanalyse 2001

34. Seeger, Norbert
International Accounting Standards (IAS) 2001

33. Moormann, Jürgen / Stehling, Frank
Strategic Positioning of E-Commerce Business Models in the Portfolio of Corporate Banking 2001

32. Sokolovsky, Zbynek / Strohhecker, Jürgen
Fit für den Euro, Simulationsbasierte Euro-Maßnahmenplanung für Dresdner-Bank-Geschäftsstellen 2001

31. Roßbach, Peter
Behavioral Finance - Eine Alternative zur vorherrschenden Kapitalmarkttheorie? 2001

30. Heidorn, Thomas / Jaster, Oliver / Willeitner, Ulrich
Event Risk Covenants 2001

29. Biswas, Rita / Löchel, Horst
Recent Trends in U.S. and German Banking: Convergence or Divergence? 2001

28. Eberle, Günter Georg / Löchel, Horst
Die Auswirkungen des Übergangs zum Kapitaldeckungsverfahren in der Rentenversicherung auf die Kapitalmärkte 2001

27. Heidorn, Thomas / Klein, Hans-Dieter / Siebrecht, Frank
Economic Value Added zur Prognose der Performance europäischer Aktien 2000

26. Cremers, Heinz
Konvergenz der binomialen Optionspreismodelle gegen das Modell von Black/Scholes/Merton 2000

25. Löchel, Horst
Die ökonomischen Dimensionen der ‚New Economy' 2000

24. Frank, Axel / Moormann, Jürgen
Grenzen des Outsourcing: Eine Exploration am Beispiel von Direktbanken 2000

23. Heidorn, Thomas / Schmidt, Peter / Seiler, Stefan
Neue Möglichkeiten durch die Namensaktie 2000

22.	Böger, Andreas / Heidorn, Thomas / Graf Waldstein, Philipp Hybrides Kernkapital für Kreditinstitute	2000
21.	Heidorn, Thomas Entscheidungsorientierte Mindestmargenkalkulation	2000
20.	Wolf, Birgit Die Eigenmittelkonzeption des § 10 KWG	2000
19.	Cremers, Heinz / Robé, Sophie / Thiele, Dirk Beta als Risikomaß - Eine Untersuchung am europäischen Aktienmarkt	2000
18.	Cremers, Heinz Optionspreisbestimmung	1999
17.	Cremers, Heinz Value at Risk-Konzepte für Marktrisiken	1999
16.	Chevalier, Pierre / Heidorn, Thomas / Rütze, Merle Gründung einer deutschen Strombörse für Elektrizitätsderivate	1999
15.	Deister, Daniel / Ehrlicher, Sven / Heidorn, Thomas CatBonds	1999
14.	Jochum, Eduard Hoshin Kanri / Management by Policy (MbP)	1999
13.	Heidorn, Thomas Kreditderivate	1999
12.	Heidorn, Thomas Kreditrisiko (CreditMetrics)	1999
11.	Moormann, Jürgen Terminologie und Glossar der Bankinformatik	1999
10.	Löchel, Horst The EMU and the Theory of Optimum Currency Areas	1998
09.	Löchel, Horst Die Geldpolitik im Währungsraum des Euro	1998
08.	Heidorn, Thomas / Hund, Jürgen Die Umstellung auf die Stückaktie für deutsche Aktiengesellschaften	1998
07.	Moormann, Jürgen Stand und Perspektiven der Informationsverarbeitung in Banken	1998
06.	Heidorn, Thomas / Schmidt, Wolfgang LIBOR in Arrears	1998
05.	Jahresbericht 1997	1998
04.	Ecker, Thomas / Moormann, Jürgen Die Bank als Betreiberin einer elektronischen Shopping-Mall	1997
03.	Jahresbericht 1996	1997
02.	Cremers, Heinz / Schwarz, Willi Interpolation of Discount Factors	1996
01.	Moormann, Jürgen Lean Reporting und Führungsinformationssysteme bei deutschen Finanzdienstleistern	1995

FRANKFURT SCHOOL / HFB – WORKING PAPER SERIES
CENTRE FOR PRACTICAL QUANTITATIVE FINANCE

No.	Author/Title	Year
20.	Reiswich, Dimitri / Wystup, Uwe FX Volatility Smile Construction	2009
19.	Reiswich, Dimitri / Tompkins, Robert Potential PCA Interpretation Problems for Volatility Smile Dynamics	2009
18.	Keller-Ressel, Martin / Kilin, Fiodar Forward-Start Options in the Barndorff-Nielsen-Shephard Model	2008
17.	Griebsch, Susanne / Wystup, Uwe On the Valuation of Fader and Discrete Barrier Options in Heston’s Stochastic Volatility Model	2008

16.	Veiga, Carlos / Wystup, Uwe Closed Formula for Options with Discrete Dividends and its Derivatives	2008
15.	Packham, Natalie / Schmidt, Wolfgang Latin hypercube sampling with dependence and applications in finance	2008
14.	Hakala, Jürgen / Wystup, Uwe FX Basket Options	2008
13.	Weber, Andreas / Wystup, Uwe Vergleich von Anlagestrategien bei Riesterrenten ohne Berücksichtigung von Gebühren. Eine Simulationsstudie zur Verteilung der Renditen	2008
12.	Weber, Andreas / Wystup, Uwe Riesterrente im Vergleich. Eine Simulationsstudie zur Verteilung der Renditen	2008
11.	Wystup, Uwe Vanna-Volga Pricing	2008
10.	Wystup, Uwe Foreign Exchange Quanto Options	2008
09.	Wystup, Uwe Foreign Exchange Symmetries	2008
08.	Becker, Christoph / Wystup, Uwe Was kostet eine Garantie? Ein statistischer Vergleich der Rendite von langfristigen Anlagen	2008
07.	Schmidt, Wolfgang Default Swaps and Hedging Credit Baskets	2007
06.	Kilin, Fiodor Accelerating the Calibration of Stochastic Volatility Models	2007
05.	Griebsch, Susanne/ Kühn, Christoph / Wystup, Uwe Instalment Options: A Closed-Form Solution and the Limiting Case	2007
04.	Boenkost, Wolfram / Schmidt, Wolfgang M. Interest Rate Convexity and the Volatility Smile	2006
03.	Becker, Christoph/ Wystup, Uwe On the Cost of Delayed Currency Fixing	2005
02.	Boenkost, Wolfram / Schmidt, Wolfgang M. Cross currency swap valuation	2004
01.	Wallner, Christian / Wystup, Uwe Efficient Computation of Option Price Sensitivities for Options of American Style	2004

HfB – Sonderarbeitsberichte der HfB - Business School of Finance & Management

No.	Author/Title	Year
01.	Nicole Kahmer / Jürgen Moormann Studie zur Ausrichtung von Banken an Kundenprozessen am Beispiel des Internet (Preis: € 120,--)	2003

Printed edition: € 25.00 + € 2.50 shipping

Download:
Working Paper: http://www.frankfurt-school.de/content/de/research/Publications/list_of_publication0.html
CPQF: http://www.frankfurt-school.de/content/de/research/quantitative_Finance/research_publications.html

Order address / contact
Frankfurt School of Finance & Management
Sonnemannstr. 9 – 11 ▪ D – 60314 Frankfurt/M. ▪ Germany
Phone: +49 (0) 69 154 008 – 734 ▪ Fax: +49 (0) 69 154 008 – 728
eMail: m.biemer@frankfurt-school.de
Further information about Frankfurt School of Finance & Management
may be obtained at: http://www.frankfurt-school.de